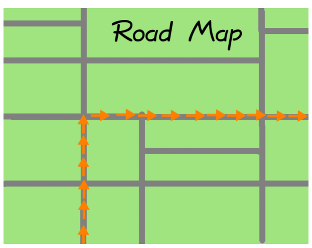
Module 0.1: Scientific Notation and Significant Digits

Numbers in the real world are ugly. They are not going to be round numbers like 20, 10, 5, or 100. Real-world numbers have decimal points and many different digits. While your algebra classes might have used coefficients like 20, 10, 5, or 100, that's not possible in this textbook, because we are concerned with real-world phenomena—particularly those that occur in business, in industry, in commerce, and in finance.

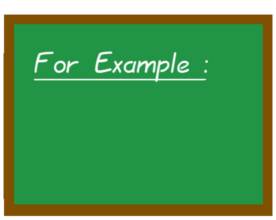
Learning to deal with real-world numbers, and all their ugliness, is a shock for many students who start this course. However, you simply have to get used to it, because the real world is not as simple and clean cut as your algebra homework once was.



Luckily for you, we have some simple tools to help make that transition smoother. I wrote this module 9 years and 1 month after my first attempt at teaching this course ended. Using all that experience, of having taught this course semester after semester, I have chosen a few key concepts to get you started safely and immediately.

First, we should know how to convert from scientific notation to ordinary notation, and vice versa. Second, we must understand how the "theory of significant digits" works as a tool for talking about the precision of a written number. (I think you'll find I have written a very easy to understand description of the theory.)

Third, you might recall from the preface that this textbook uses six significant digits for almost all problems. I'll spend a little bit of time explaining why I made that choice. Fourth, we'll talk about how to compare two calculations, in light of the theory of significant digits. Fifth, I will explain the "golden rule of accuracy"—never round (or truncate) in the middle of a problem, only at the end of a problem—and then give you several examples of why this is true.



0-1-1

Before we even begin to discuss scientific notation, it is worthwhile to discuss why we need it in business, economics, or finance. For example, if you are working on a compound interest problem, and have a *nominal rate* of

$$r = 1\frac{1}{4}\% = 0.0125$$

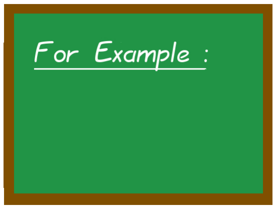
compounded daily (using a 360-day banker's year), and are asked to compute the *periodic rate*, then it turns out that you'd do this by computing $0.0125 \div 360$ on your calculator. (If you don't know what the difference is between a nominal rate and a periodic rate, or why we're using 360 instead of 365 or 366, don't worry about that right now. We'll learn about that soon enough, in the module "The Basics of Compound Interest.")

Whether you use a hand-calculator or MS-Excel, you'll get an answer that looks something like

$$0.0125 \div 360 = \texttt{3.4722222E-05}$$

and you need to know scientific notation in order to understand what that means. It turns out that this means

$$3.4722222 \times 10^{-5} = 0.000034722222222 = 0.00003472\overline{2}$$



0-1-2

The previous box established that we need to know about scientific notation to handle very small numbers. It turns out that we also need to know about scientific notation to handle very large numbers, especially if we're going to have an intelligent conversation about our nation's economy. According to the website

`https://www.treasurydirect.gov/NP/debt/current`

the US Federal Debt on January 15th, 2017, was

$$\$\ 19,941,807,383,847.05$$

while the US population, according to the website `http://www.census.gov/popclock/`, happens to have been

$$324,382,823$$

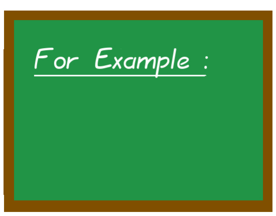
Depending on your model of calculator, you might or might not be able to enter 324,382,823 directly without scientific notation. However, for almost all calculators, you would need to enter 19,941,807,383,847.05 with scientific notation. Even with MS-Excel, where you could enter 19,941,807,383,847.05 directly without scientific notation, it will convert that number into scientific notation immediately, showing

`1.9941807E13`

or perhaps one digit more or less. (It depends on how wide the column is, in case you are curious.)

In any case, to work with these large numbers, we need to know scientific notation.

We're now going to learn about how to convert to and from scientific notation. There is a slow way to do this, and a fast way to do it. Many of students were only shown the slow way in school, and that's why they imagine that this subject is tedious. In reality, you'll see that this is all rather quick.

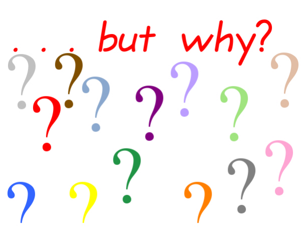


0-1-3

Suppose that you have to convert 3.28×10^{-4} from scientific notation into ordinary notation. Most of us know that the -4 means that we'll be shifting the decimal point four spots. As it turns out, negative exponents mean a shift to the left, and positive exponents mean a shift to the right, so we'll be shifting to the left.

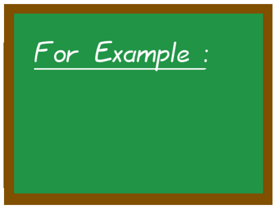
Doing this one step at a time is the slow way.

$$\begin{array}{crcll} & 3.28 & \times & 10^{-4} & \text{Ready!} \\ = & 0.328 & \times & 10^{-3} & \text{first move complete} \\ = & 0.0328 & \times & 10^{-2} & \text{second move complete} \\ = & 0.00328 & \times & 10^{-1} & \text{third move complete} \\ = & 0.000328 & \times & 10^{0} & \text{fourth move complete} \\ = & 0.000328 & \times & 1 & = 0.000328 \end{array}$$



This is undoubtedly correct, but it is also a lot of work. The above method is probably what inspired scientific notation in the first place. (Well, that and some technicalities about logarithm tables that would be very tedious and boring to explain.)

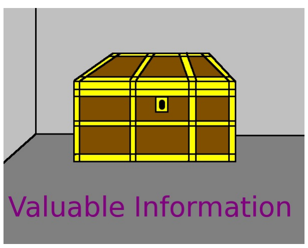
The method in the previous box never fails, but it is just too slow. Soon enough, I'll show you what real scientists do. Meanwhile, let's see one more theoretical example.



0-1-4

This will give you an idea of why I call this a "slow" method. Consider changing 4.98×10^9 from scientific notation into ordinary notation. Recall, negative exponents mean a shift to the left, and positive exponents mean a shift to the right. We will be shifting to the right this time, since 9 is positive here.

$$\begin{array}{crcll}
 & 4.98 & \times & 10^9 & \text{Ready!} \\
= & 49.8 & \times & 10^8 & \text{first move complete} \\
= & 498 & \times & 10^7 & \text{second move complete} \\
= & 4980 & \times & 10^6 & \text{third move complete} \\
= & 49,800 & \times & 10^5 & \text{fourth move complete} \\
= & 498,000 & \times & 10^4 & \text{fifth move complete} \\
= & 4,980,000 & \times & 10^3 & \text{sixth move complete} \\
= & 49,800,000 & \times & 10^2 & \text{seventh move complete} \\
= & 498,000,000 & \times & 10^1 & \text{eighth move complete} \\
= & 4,980,000,000 & \times & 10^0 & \text{ninth move complete} \\
= & 4,980,000,000 & \times & 1 & = 4,980,000,000
\end{array}$$



By the way, it is worthwhile to mention that in scientific notation, we always have only one digit to the left of the decimal point—never more than one digit. While

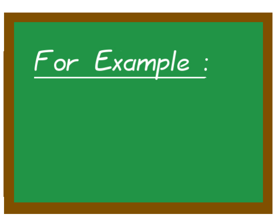
$$3.28 \times 10^{-4} \text{ and } 4.98 \times 10^9$$

are in scientific notation, the numbers

$$11.24 \times 10^{-3} \text{ and } .987 \times 10^{12}$$

are not. The 11.24×10^{-3} has two digits to the left of the decimal point, and the $.987 \times 10^{12}$ has no digits to the left of the decimal point.

Moreover, while the digit to the left of the decimal point can be 1, 2, 3, 4, 5, 6, 7, 8, or 9, it cannot be a zero. You will see why, shortly.



0-1-5

Now, I'm going to show you the shortcut. When a number is very small and in scientific notation, the exponent is negative. Reconsider the number

$$3.28 \times 10^{-4} \leftarrow \text{See, the exponent is four.}$$

which must become

$$\underbrace{0.000}_{\text{four zeros}}\quad 328 = 0.000328$$

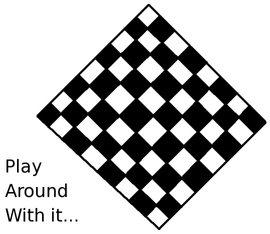
Likewise, let's look at

$$1.25 \times 10^{-3} \leftarrow \text{See, the exponent is three.}$$

must now become

$$\underbrace{0.00}_{\text{three zeros}}\quad 125 = 0.00125$$

As you can see, for a negative exponent, the number of zeroes is equal to the (absolute value of) the exponent. If you see -3 up there, there should be three zeroes, and if you see -4 up there, there should be four zeroes. However, be certain to include that zero found to the left of the decimal point.

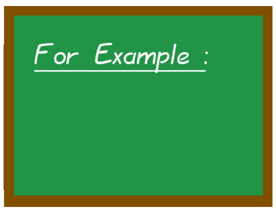


\# 0-1-6

Convert the following numbers from scientific notation to ordinary notation.

- Convert 5.281×10^{-3}.
- Convert 3.491×10^{-4}.
- Convert 6.821×10^{-2}.
- Convert 9.807×10^{-1}.

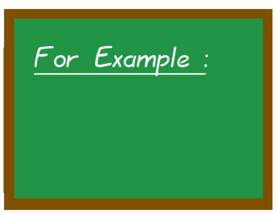
The answers will be given on Page 49 of this module.



\# 0-1-7

The shortcut also works for very large numbers. When a number is very large and in scientific notation, the exponent is positive. First, consider 7×10^5. We're just going to slap five zeros after that 7. We obtain

$$7 \underbrace{00000}_{\text{five zeros}} = 700000 = 700,000$$



\# 0-1-8

The shortcut method gets more interesting if there is a decimal point. Consider,

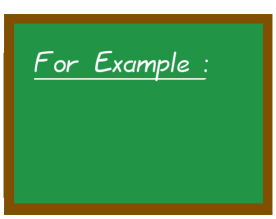
$$4.98 \times 10^9$$

which first becomes

$$498 \times 10^7$$

as the two decimal places "eat up" two of the nine leaving seven behind. Then you can write the following:

$$498 \underbrace{0000000}_{\text{seven zeros}} = 4980000000 = 4,980,000,000$$



\# 0-1-9

One more example of the shortcut method with a decimal is

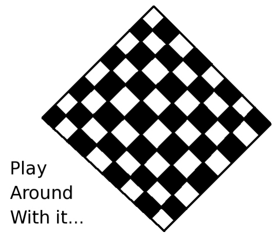
$$5.2148 \times 10^{12}$$

which should become

$$52148 \times 10^8$$

as the four decimal places "eat up" four of the twelve, leaving eight behind. Then you can write the following:

$$52148 \underbrace{00000000}_{\text{eight zeros}} = 5214800000000 = 5,214,800,000,000$$

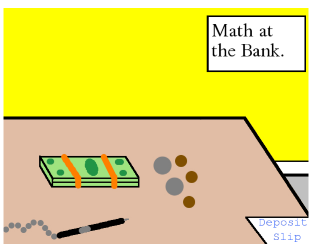


0-1-10

Convert the following numbers from scientific notation to ordinary notation.

- Convert 9.724×10^6.
- Convert 4.127×10^8.
- Convert 176×10^7. (Technically, this is not scientific notation, but convert it to ordinary notation anyway.)
- Convert 5.68912×10^4.

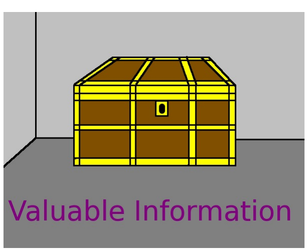
The answers will be given on Page 49 of this module.



One of the most horrifying mistakes that a student can make is failing to grasp the difference between a million, a billion, and a trillion. Getting those mixed up is extremely embarrassing. It definitely would be a terminal error during a job interview.

Certainly, discussions of millions come up in any business except the smallest family businesses. Discussions of billions are extremely common in political discourse, especially about budgets. Naturally, it is required that one discuss trillions when talking of the federal debt.

Luckily, I've only seen about one student every three years making this sort of error. Just for completeness, here is a reference table for you.



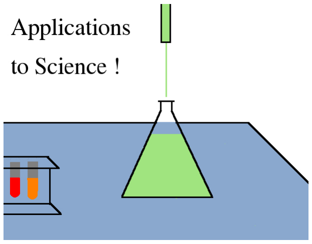
These are the names of large powers of ten.

Number	Name	Power of Ten
1,000	"one thousand"	10^3
1,000,000	"one million"	10^6
1,000,000,000	"one billion"	10^9
1,000,000,000,000	"one trillion"	10^{12}
1,000,000,000,000,000	"one quadrillion"	10^{15}
1,000,000,000,000,000,000	"one quintillion"	10^{18}

There are others, such as "octillion" for 10^{27}, but those are essentially never used. Even quintillion is almost never seen, though quadrillion does come up once in a rare while. (By the way, this is the American usage. Some other countries, including the UK, have a different naming scheme entirely.)

I wanted to double check the definition of octillion, so I looked it up on the internet. I was anticipating that this number almost never appears on the net other than in encyclopedias and dictionaries, but then I found the word "octillion" in the title of an article published in *Fortune Magazine*!

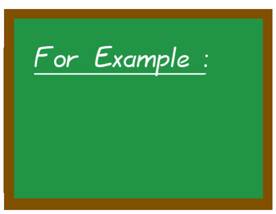
The British energy company Ovo performed a computation, and figured out the cost to run The Death Star from the movie series *Star Wars*. The extremely interesting article was titled "The Death Star Would Cost \$ 7.8 Octillion a Day to Run." by David Z. Morris, published on December 3rd, 2016.



Here's one way to understand the million/billion/trillion relationship:

- A thousand seconds is $1000/60 = 16.6666\cdots$ minutes.
- Likewise, a million seconds is $16,666.6\cdots/60 = 277.777\cdots$ hours, which is also $277.777\cdots/24 = 11.5740\cdots$ days.
- Next, a billion seconds is $11,574.0\cdots/365.2422 = 31.6887\cdots$ years.
- Clearly, a trillion seconds is then $31,688.7\cdots$ years. That's longer than recorded history. Looking back 31,688 years would put you deep into The Stone Age.
- Therefore a quadrillion seconds is $31.6887\cdots$ million years, before anatomically modern human beings, or their recent ancestors, existed.
- Last but not least, a quintillion seconds is just slightly less than 32 billion years, which is much older than the current estimated age of the universe, around 13.8 billion years.

Now I think you can see why someone should be horrified when a student says "Whatever! A billion and a trillion are basically the same thing!" That is because it is absurd to compare 31.688 years to 31,688 years. Clearly, these are not the same thing.

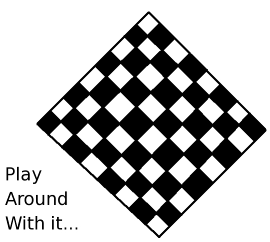


A student once asked me, during a test, if I could tell her how she should put 12.67 million into her calculator. This made me understand that there are some students who do not understand what "12.67 million" actually means.

It is really easy, as it turns out. Because "million" means 10^6, then we can write

$$\text{“12.67 million”} = 12.67 \times 10^6 = 1267 \times 10^4 = 1267\underbrace{0000}_{\text{four zeros}} = 12,670,000$$

0-1-11

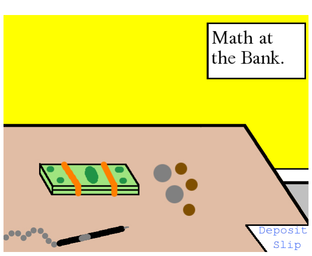


Write the following in both scientific and ordinary notation:

- 4.8 billion dollars
- 16.5 million dollars
- half a billion dollars
- 19.94 quadrillion Japanese Yen

The answers will be given on Page 49 of this module.

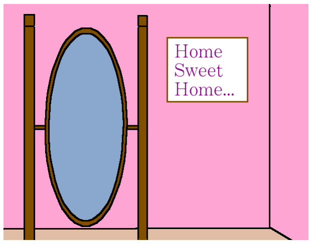
0-1-12



Some students would scoff at the entry "19.94 quadrillion Japanese Yen" in the previous box. If you are curious, 19.94 quadrillion Japanese Yen is indeed an impossibly large amount of money. Those students are correct to scoff.

For comparison, the national debt of Japan, on January 24th, 2017, was 1.02621 quadrillion Japanese Yen, which is $9.01734\cdots$ trillion dollars. (So if anyone ever asks you whether the word "quadrillion" ever comes up in business, now you have your answer.) That's much smaller than the US national debt, which was $19.9473\times$ trillion dollars on the same day.

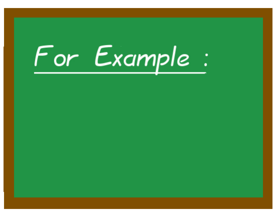
If you are curious, here are the websites that I used to find this out.
`http://www.nationaldebtclocks.org/debtclock/japan`
`http://www.nationaldebtclocks.org/debtclock/unitedstates`



A Pause for Reflection...

Once, when I was teaching this module, a student in the front row had her face in an expression of recoil or disgust and exclaimed "This is chemistry!"

It might have been the case that it was in high-school chemistry that you first saw significant digits; or perhaps it was in high-school physics. It might have been the case that you first saw scatter plots in your high-school chemistry, physics, or ecology classroom. Nonetheless, rest assured that significant figures, scientific notation, scatter plots, and other tools, are legitimate parts of applied mathematics—in fact, they are indispensable for approaching real-world problems.



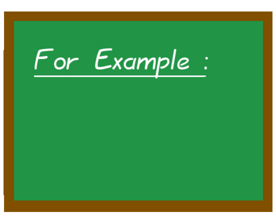
0-1-13

Let's imagine that we are asked to convert 0.000512 into scientific notation.

Usually the way that this is taught is that you start with the number, and shift the decimal point one spot at a time, until you have only a single digit to the left of the decimal point. The number of shifts will be the exponent in scientific notation. For our case,

$$\begin{array}{rrcll} & 0.000512 & \times & 10^{0} & \text{Ready!} \\ = & 0.00512 & \times & 10^{-1} & \text{first move complete} \\ = & 0.0512 & \times & 10^{-2} & \text{second move complete} \\ = & 0.512 & \times & 10^{-3} & \text{third move complete} \\ = & 5.12 & \times & 10^{-4} & \text{fourth move complete} \end{array}$$

This seems a bit slow. Momentarily, we'll see that the shortcut method is much more efficient



0-1-14

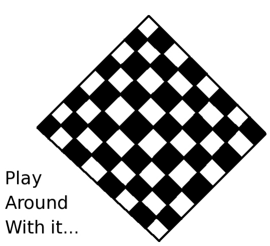
Let us suppose that I am asked to convert 0.000512 and 0.00625 into scientific notation.

This is easy to do with the shortcut. I merely need to count the leading zeros—that will be the exponent.

$$0.000512 = \underbrace{0.000}_{\text{four zeros}} \; 512 = 5.12 \times 10^{-4}$$

Similarly,

$$0.00625 = \underbrace{0.00}_{\text{three zeros}} \; 625 = 6.25 \times 10^{-3}$$

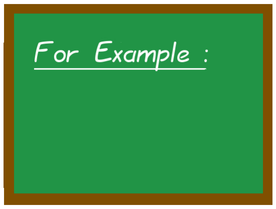


0-1-15

Please convert the following numbers into scientific notation.

- Convert 0.7834
- Convert 0.002356
- Convert 0.0123
- Convert 0.0008754

The answers will be give on Page 49 of this module.



0-1-16

As you can see, the shortcut is great for converting small numbers into scientific notation. You might wonder about longer numbers. Let us suppose that I am asked to convert 36,701.89 and 4,980,000,000 into scientific notation.

This is also easy to do with the shortcut. I merely need to count the digits between the decimal point, and the leading digit, but leaving the leading digit out. Consider

$$36,701.89 = 3\underbrace{6,701}_{\text{4 digits}}.89 = 3.670189 \times 10^4$$

as well as

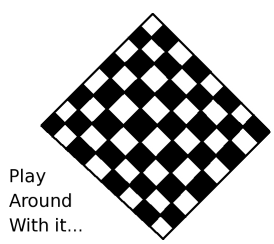
$$4,980,000,000 = 4,\underbrace{980,000,000}_{\text{9 digits}} = 4.98 \times 10^9$$

It might be good to verify that the shortcut actually produces the same answer as doing the problem the long way.

Consider the first number from the previous example.

	36,701.89	$\times$	10^0	Ready!
=	3670.189	$\times$	10^1	first move complete
=	367.0189	$\times$	10^2	second move complete
=	36.70189	$\times$	10^3	third move complete
=	3.670189	$\times$	10^4	fourth move complete

As you can see, we got the same answer.

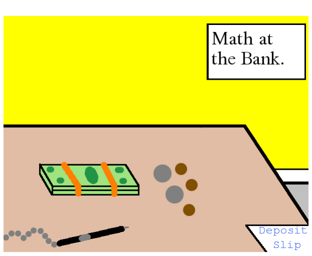


0-1-17

The GDPs and populations of several nations are listed below. These are from the *CIA World Factbook*, and represent mid-2009 estimates. They are the G-8 nations, which comprise roughly 50% of the world GDP.

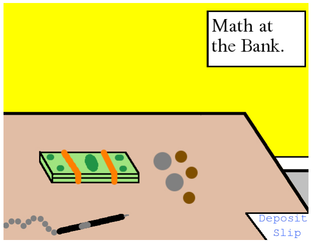
Nation	Gross Domestic Product	Population
Canada	\$1,319,000,000,000	33,487,208
France	\$2,635,000,000,000	64,057,792
Germany	\$3,235,000,000,000	82,329,758
Italy	\$2,090,000,000,000	58,126,212
Japan	\$5,049,000,000,000	127,078,679
Russia	\$1,232,000,000,000	140,041,247
The United Kingdom	\$2,198,000,000,000	61,113,205
The United States	\$14,270,000,000,000	307,212,123

Convert the GDPs into scientific notation. The answers will be given on Page 50.



Depending on how much economics you've had, you might be wondering what a GDP is, or what the G-8 nations are. When you first enter a new subject, there's a ton of vocabulary that you'll simply have to acquire. I'll do my best throughout this textbook to define my terms and give the real-world economic or financial context. However, sometimes I might forget. You live in the age of the internet! Look things up! (You'll have a huge advantage over your classmates if you do this, because you'll walk into the intermediate and advanced classes with a larger-than-expected vocabulary and familiarity.)

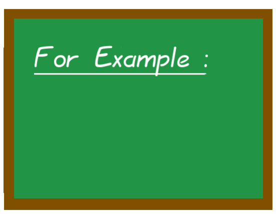
It is a good idea to keep a good website (such as Wikipedia, or even better, Investopedia) bookmarked, so that these look-ups are instantaneous. I'll let you look up GDP and the G-8 nations yourself, now.



You've probably seen some very simple ordinary calculators, that have only the four basic operations (addition, subtraction, multiplication, division) and maybe percentages. These simple "four-function" calculators were once far more common, back when more powerful calculators were somewhat expensive. Bank tellers and even entry-level businessmen would often only have a four-function calculator. Those calculators usually had 8 or 9 digits of display, as it turns out.

There was another application of scientific notation back then. You could perform operations upon very large numbers, even with such a simple calculator, if you were savvy with scientific notation. You could even work with numbers that were too large to even enter into the calculator.

When I first started teaching *Finite & Financial Mathematics*, I used to teach this extra trick as part of the course. However, I've come to realize that even the calculators built into our phones today are so powerful, that this trick is no longer important. Therefore, I'm going to show you just one example of it (in case you are curious), and move onward.



0-1-18

A few boxes ago, we learned that the United States has 1.427×10^{13} dollars per year as its GDP, and a population of $3.07212\cdots \times 10^8$ people. Suppose we want to compute the GDP per capita. (Note that *per capita* is Latin for "per head", which translates figuratively to "per person.")

We only need to compute

$$\left(1.427 \times 10^{13}\right) \div \left(3.07212\cdots \times 10^8\right)$$

which should not be too hard. Yet, suppose we only had a four-function calculator. What would we do in that case? We cannot even enter a 13 digit number into such a simple calculator.

First, you'd compute

$$1.427 \div 3.07212\cdots = 0.464500$$

on the calculator. Then, because $13 - 8 = 5$, you'd adjust by multiplying by 10^5.

$$0.464500 \times 10^5 = 46,450$$

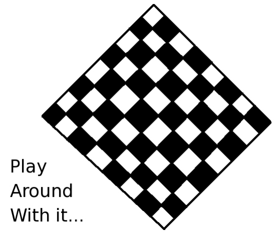
This seems right, as it claims that \$ 46,450 would be the productivity of a randomly chosen US resident in mid-2009. Of course, a randomly chosen resident might be 2 years old, 95 years old, unemployed or in prison, so it is slightly lower than the average wages of a random worker.

By the way, in the previous box, we subtracted the exponents from the scientific notation because we are dividing. If we were multiplying, we would add, instead.

Though you wouldn't actually write all this out, what we were really doing was

$$\frac{1.427 \times 10^{13}}{3.07212\cdots \times 10^8} = \frac{1.427}{3.07212\cdots} \times \frac{10^{13}}{10^8} = 0.464500 \times 10^5 = 46,450$$

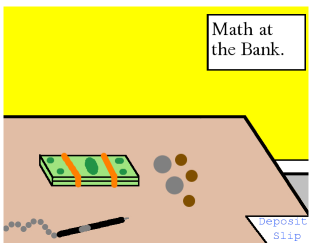
If you are still confused about $10^{13} \div 10^8 = 10^5$, then note it will be explained in the module "How Exponents Really Work."



0-1-19

I'd like you to now compute the GDP per capita for Russia and for France. You can use the trick from the previous example if you like, or you can compute it any other way that you want.

- What is the GDP per capita (in mid-2009) for Russia? [Answer: \$ 8797.40.]
- What is the GDP per capita (in mid-2009) for France? [Answer: \$ 41,134.73.]

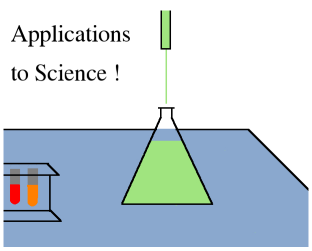


At this point, we've finished up with scientific notation, and I'd like to take a moment and talk about precision. Let's say that you are working for an asphalt factory, and you've performed a standard computation to determine how much gravel to order. Your final answer, on your calculator, is

`14.1393188`

metric tons of gravel.

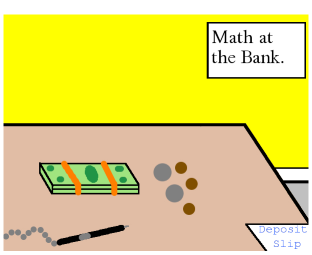
This is not the number that your company is going to give to the gravel supplier. To grasp this, look at that last 8. It is in the seventh decimal place, so that last 8 actually represents 8×10^{-7} metric tons of gravel. In the next box, we're going to try to wrap our minds around that number.



In the previous box, we saw the number 14.1393188 and we're trying to understand how absurd that last 8 is, by explaining what 8×10^{-7} metric tons of gravel actually means.

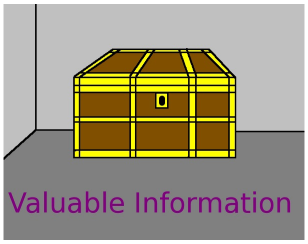
- You might not be accustomed to working with metric tons, but a metric ton is 1000 kg. That means the last 8 represents 8×10^{-4} kg of gravel.
- Since a kilogram is 1000 g, that represents $8 \times 10^{-1} = 0.8$ g of gravel.
- This can also be written as 800 mg of gravel.
- That's rather close to 1/3rd the weight of a US penny or US dime.

Now imagine this. Your company is ordering "14 point something" metric tons of gravel. Can anyone possibly care about a small bit, weighing less than a penny?! Is there a scale accurate enough in order to measure a delivery of over 14 tons, accurate to the hundreds of milligrams?!



Now we've established that excessive accuracy can make you look very silly in the workplace. Of course, it isn't right to say "please deliver somewhere around 10 to 20 metric tons of gravel" either.

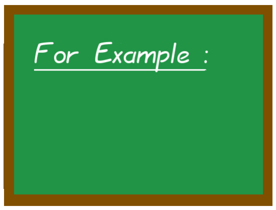
What this really means is that we need a precise way of describing and discussing the relative precision of numbers. Whenever a question ends in a real number, we have to say how much precision is required. The theory of significant digits (sometimes called significant figures) will address exactly this point.



Here are the rules for significant figures:

- Any non-zero digit: 1, 2, 3, 4, 5, 6, 7, 8, or 9, is *always* significant.
- A zero between two non-zero digits is *always* significant.
- Trailing zeros, the zeros at the end of a large number, are *never* significant.
- Leading zeros, the zeros at the start of a number between -1 and 1, are *never* significant.

Examples will make this far more clear. I have four of them for you.

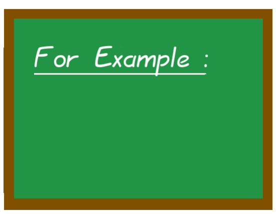


0-1-20

Consider the number 0.000300471. Let's mark the digits according to the rules of significant figures.

					1^{st}		3^{rd}		5^{th}	
					↓		↓		↓	
0	.	0	0	0	3	0	0	4	7	1
						↑		↑		↑
						2^{nd}		4^{th}		6^{th}

As you can see, this number has six significant digits.

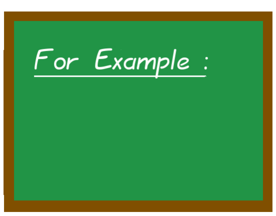


0-1-21

Consider the number 541.29. Let's mark the digits according to the rules of significant figures.

1^{st}		3^{rd}			5^{th}
↓		↓			↓
5	4	1	.	2	9
	↑			↑	
	2^{nd}			4^{th}	

As you can see, this number has five significant digits.

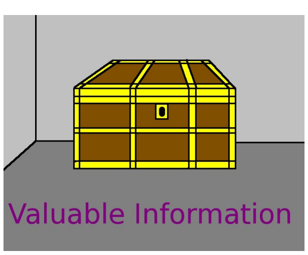


0-1-22

Consider the number 48,700,000. Let's mark the digits according to the rules of significant figures.

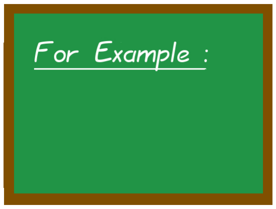
1^{st}			3^{rd}						
↓			↓						
4	8	,	7	0	0	,	0	0	0
	↑								
	2^{nd}								

As you can see, this number has three significant digits.



Perhaps it is a good time to mention that the first significant digit is sometimes called the *most significant digit*, and the last significant digit is sometimes called the *least significant digit*.

Think about it in terms of money: if you win a lotto totaling \$ 123,456,789, then the "1," representing \$ 100,000,000, is the most important digit in that number.

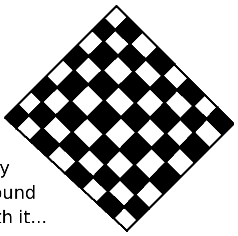


0-1-23

Consider the number 14.1393188, from our conversation about gravel on Page 38. Let's mark the digits according to the rules of significant figures.

1^{st}			3^{rd}		5^{th}		7^{th}		9^{th}
↓			↓		↓		↓		↓
1	4	.	1	3	9	3	1	8	8
	↑			↑		↑		↑	
	2^{nd}			4^{th}		6^{th}		8^{th}	

As you can see, this number has nine significant digits, which is pretty crazy. Few instruments can measure weight to that degree of accuracy.



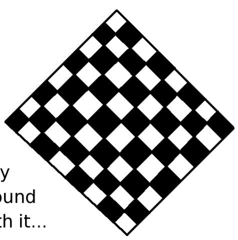
Play Around With it...

0-1-24

Tell me how many significant digits the following numbers have:

- 13,768,000
- 241.589
- 0.005600198
- 0.0003125
- 17,800,000,000

The answers will be given on Page 50 of this module.



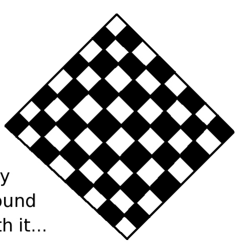
Play Around With it...

0-1-25

Identify the following numerals:

- What is the 2nd significant digit of 17,800,000,000?
- What is the 3rd significant digit of 0.0003125?
- What is the 4th significant digit of 241.589?
- What is the 5th significant digit of 13,768,000?
- What is the 6th significant digit of 0.005600198?

[Answer: 7, 2, 5, 8, and 9.]



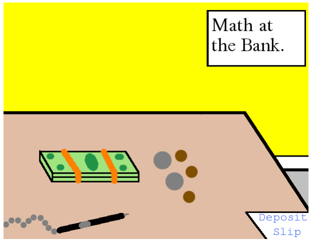
Play Around With it...

0-1-26

Identify the following numerals:

- What is the 2nd significant digit of 39,100,000,000?
- What is the 3rd significant digit of 0.0004879?
- What is the 4th significant digit of 451.327?
- What is the 5th significant digit of 28,974,000?
- What is the 6th significant digit of 0.007200973?

[Answer: 9, 7, 3, 4, and 7.]

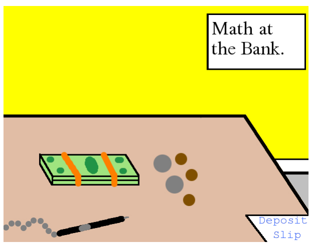


Now that we have acquired a measurement for how precise a number is, the next natural question to ask is clearly "How much precision should you use?" Based on my working career, prior to going into teaching, this is what I can tell you.

Most engineering problems used 4 or 5 digits of accuracy, with 4 being far more common than 5. However, Civil Engineering (perhaps because they work in mud, steel, wood, and concrete) would use 3 digits. There were exceptions, of course. Aeronautical and astronautical computations often had a tremendous degree of precision.

In the financial industry, four digits is very common. For example, percentages are often reported in the form of 56.78%. That 8 represents 1% of 1%. There's even a name for that—a basis point is 1% of 1%. So if someone says that the fed is going to raise the interest rate 25 basis points, that's 25% of 1% or 1/4th of 1%.

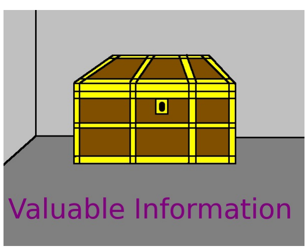
We will continue in the next box.



Continuing with the previous box, we were discussing how many significant digits are used in the real world, for various workplaces.

It is key to understand that in any workplace, you should always ask your teammates what to do, and do what they tell you—especially at the start of your career. That's one nice thing about the switch to team-based management which started in the 1980s and 1990s. When you are new on the job, you are surrounded by people who have a vested interest in ensuring that you prosper.

For example, one of my former students, who is a proofreader of this textbook in his spare time, works for a mid-sized bank in Wisconsin and they report percentages to the tenth of a percent. In such situations, the new employee should say "Sure, let's do it that way," and comply without hesitation.



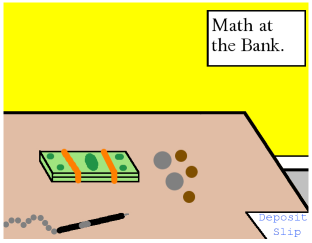
However, in this textbook, I almost always use six digits of accuracy. Sometimes, we really want a lot of accuracy to demonstrate a cool mathematical phenomenon, and so I'll switch to nine digits of accuracy. That doesn't happen very often, and I will point it out very clearly when it is needed.

Throughout this textbook, almost all problems will ask for, and deliver, six significant digits. Now I'll take a moment or two to explain why.

The reason for this is that rounding error is a persistent problem in computing actual mathematical answers, whether with a hand calculator, a computer, or a supercomputer—or even with logarithm tables, which were used prior to the invention of hand calculators and computers. Math majors often take an entire semester—if not a year—of *Numerical Analysis*, which consists of learning advanced algorithms (advanced ways to compute things) so as to minimize rounding error.

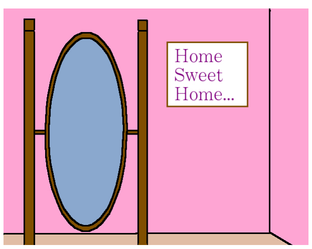
Usually, one needs a course in differential equations, or a course in matrix algebra, if not both, plus three semesters of calculus, in order to understand *Numerical Analysis*. Therefore, I cannot go into those details here, because you wouldn't be able to understand that at all.

However, you can trust the rule of thumb—long established in many branches of applied mathematics, physics, and engineering—that you should have at least two "spare" significant digits. Therefore, if you need 3, then use at least 5; if you need 4, use at least 6. That was my original reason for choosing six significant figures.



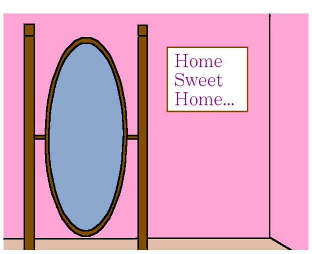
Instead, here is a practical example of why it was useful for me to have chosen six significant figures as the standard of accuracy for this textbook.

- Consider an investment of \$ 5000 at $2\frac{1}{2}\%$ interest, compounded quarterly, over a period of four years. The final amount turns out to be \$ 5524.13.
- If a student compounds monthly, instead of quarterly, because their accounting class only taught them monthly and didn't teach them quarterly, then they'll get \$ 5525.27.
- As you can see, the distinction is clear at six significant figures: \$ 5524.13 vs \$ 5525.27.
- However, if I were using four significant figures, it would be \$ 5524 vs \$ 5525, where the mistake is almost invisible. I say "almost invisible" because the numbers are still different, being off by a dollar.



A Pause for Reflection...
Another reason why it is nice to use six significant figures in a textbook, or a mathematics examination, has to do with sloppy handwriting.

- Some students write 0s that look like 6s, because they don't accurately close the circle when the loop that forms the 0 completes.
- Some students write 9s that look like 4s, because they don't close the roof of the 9 properly.
- Students in a hurry might make 5s turn into 6s, if there is not enough empty space in the lower-left-hand corner of the 5.
- Also, a 3 can turn into an 8, for those with really poor handwriting.
- On the plus side, having some extra accuracy allows me to ignore an illegible digit—if the other five are correct, then I can give the student the benefit of the doubt.
- On the flip side, students should really think about their handwriting. Writing neatly does take a little more time than writing sloppily, it is true. However, it doesn't take all that much more time to write neatly, and it certainly is less time than repeating the entire course because of low grades.



A Pause for Reflection...
All in all, I'm sure you'll see that there are some very good reasons for using six significant digits as the standard in this textbook.

- Having a uniform policy saves us from having to discuss, in every situation dispersed over more than 1000 pages of this textbook, what level of accuracy is appropriate.
- Using six significant digits is a good practice for guaranteeing accuracy in the first four significant digits.
- Certain conceptual errors might go undetected if I used fewer than six digits.
- Using six significant digits allows me to overlook some handwriting errors, in some cases.

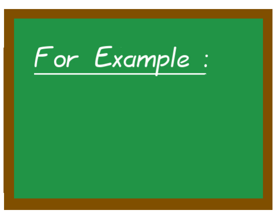
Let's say that you're working on a computation (using six significant digits), and your answer is

$$17.45876124$$

but you want to compare your answer with others. This can happen while you are doing your homework (you wish to compare your answer with the book's answer), when studying with classmates, or even in industry.

Sometimes in industry, important calculations are done in two different ways, with the hope of getting the same answer both ways, which should raise one's confidence in the answer's validity.

The theory of significant figures allows us to have a more solid concept of "close enough." We'll explore this in the next box.



0-1-27

Let's say that you have the number

$$17.45876124$$

after working on a computation (using six significant digits), and you want to compare your work with others. What numbers should be considered to be equivalent?

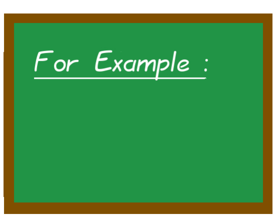
First, we should put a marker, a slash, a vertical line, or something like that between the last valid digit (the sixth significant digit, in this case) and the first invalid digit (the seventh significant digit, in this case). Now we have this

$$17.4587|6124$$

but part of the theory of significant figures is that the last significant figure is allowed to be off. It does not carry the same degree of certainty as the others.

For this reason, we should accept as equivalent any number that starts with

$$17.4586\cdots \text{ or } 17.4587\cdots \text{ or } 17.4588\cdots$$



0-1-28

The following actually happened once, when I was teaching this class. We were checking our work at the end of a problem, and we were hoping to get $ 20,000.00, but we actually got $ 19,999.99 as it turns out. To me, these are essentially the same number. However, one of the young ladies in the classroom disagreed with me a lot, saying that it is not the same.

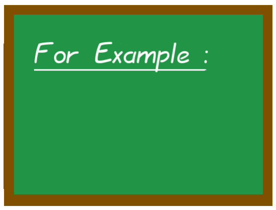
Let's follow our procedure, and see how that helps us. First, we mark the number after the sixth significant digit. We have

$$20,000.0|0$$

and this means that we should accept as equivalent

$$19,999.9\cdots \text{ or } 20,000.0\cdots \text{ or } 20,000.1\cdots$$

As you can see, there is no doubt that $ 19,999.99 falls solidly in that range.



0-1-29

One last example, and then we can move on to more interesting conversations. Let's say your final answer is \$ 28,974,000, and we are working with six significant figures. What should you accept as equivalent?

We take the number, and mark the boundary of significance,

$$28,974,0|00$$

which means that we should accept as equivalent any answers of the form

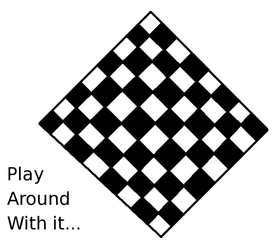
$$28,973,9?? \text{ or } 28,974,0?? \text{ or } 28,974,1??$$

where the "?" mark can be any digit at all. However, this notation looks kind of cartoonish and unprofessional.

What we do in this case is retreat back into scientific notation, and write instead

$$2.89739\cdots \times 10^7 \text{ or } 2.89740\cdots \times 10^7 \text{ or } 2.89741\cdots \times 10^7$$

which means the same thing, but looks more professional.



0-1-30

We return now to some numbers that we saw a few boxes ago. For each of these, we are working with six significant figures.

- What shall we accept as equivalent to 451.327?
  [Answer: $451.326\cdots$ or $451.327\cdots$ or $451.328\cdots$.]
- What shall we accept as equivalent to 0.007200973?
  [Answer: $0.00720096\cdots$ or $0.00720097\cdots$ or $0.00720098\cdots$.]
- What shall we accept as equivalent to 0.005600198?
  [Answer: $0.00560018\cdots$ or $0.00560019\cdots$ or $0.00560020\cdots$.]
- What shall we accept as equivalent to 0.0004879?
  [Answer: $0.000487899\cdots$ or $0.000487900\cdots$ or $0.000487901\cdots$.]

One very nice aspect of this theory is that if you have a final answer like

$$17.45876124$$

and mark the spot between the sixth and seventh significant figures, this way

$$17.4587|6124$$

then you do not have to agonize about whether to round, or to truncate.

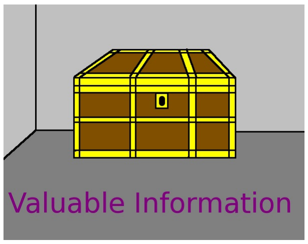
When we say *truncation*, we mean cutting the number, simply deleting everything after that vertical mark. We would write $17.4587\cdots$. Truncation is sometimes called "chopping."

When we say *rounding*, we mean looking at the first digit to be discarded. If it is a 0, 1, 2, 3, or 4, the last surviving digit is left unchanged. If it is a 5, 6, 7, 8, or 9, then those who round would increment the last surviving digit. They would write 17.4588.

We'll continue in the next box.

Continuing with the previous box, while rounding is very popular with high school teachers and accountants, there are actually some very good reasons for using truncation. For example, this textbook uses truncation (as do most university-level textbooks) for reasons that will be explained at the end of this module.

However, I'm happy to tell you that it will never matter. Whether you round or you truncate, you'll always remain inside this range of real numbers accepted as equivalent. You'll never stray outside it. So, there is nothing to worry about or debate.



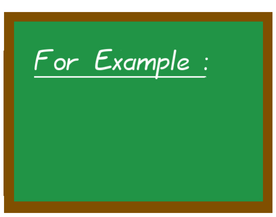
Now, we're going to discuss "the golden rule of accuracy." The rule says the following:

Never round (or truncate) in the middle of a problem; only round (or truncate) at the end of a problem.

By the way, the golden rule of accuracy is a very good rule in applied mathematics generally. However, it is particularly important in financial mathematics, because interest rates are very sensitive numbers.

When we say that interest rates are very sensitive numbers, what we mean is that a slight change in the interest rate can have catastrophic consequences to the answer. To see this, let's take two solid real-world examples.

We'll consider three students—Albert, Beatrice, and Christopher. Beatrice will always obey the golden rule of accuracy and never round in the middle of a problem. However, the boys will be disobedient, and will round during the problem, resulting in wrong answers.



\# 0-1-31

Imagine that funds are to be put away when a child is born, for his or her college education. Thus the account will run for 18 years, and suppose it has a 7% interest rate. Perhaps \$ 10,000 is deposited, and the account will compound monthly. We want to know how much is in the account at the end. (Though you might not know how to solve such a problem now, you'll learn about it soon, in the module "The Basics of Compound Interest." In fact, you'll see this problem again on Page 281.)

The interest rate per month (the periodic rate) will be $0.07/12 = 0.00583\overline{3}\cdots$. In the next box, we will see what answers these three students get.

Continuing with the previous box, Beatrice will dutifully use the exact value of i to as many decimal places as her calculator allows. Again, don't worry if you don't yet know how to perform these calculations—that will be covered in the module "The Basics of Compound Interest," which you will get to soon.

$$A = 10,000(1 + 0.005833\overline{3})^{18\times 12} = 10,000(1.005833\overline{3})^{216} = 10,000(3.51253\cdots) = 35,125.39$$

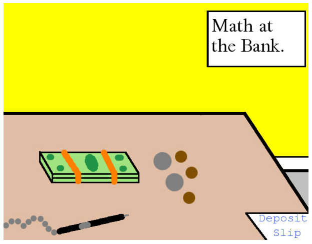
As I mentioned, the boys round off. Albert will use 0.0058.

$$A = 10,000(1 + 0.0058)^{18\times 12} = 10,000(1.0058)^{216} = 10,000(3.48748\cdots) = 34,874.85$$

Next, Christopher will use 0.006.

$$A = 10,000(1 + 0.006)^{18\times 12} = 10,000(1.006)^{216} = 10,000(3.64052\cdots) = 36,405.23$$

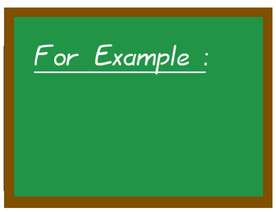
We'll discuss this further, in the next box.



Looking at the previous box, we can see that the difference is enormous.

Albert is off by \$ 250.54, but Christopher is off by \$ 1279.84. These are not small amounts, and for sure, both Albert and Christopher will be fired.

Remember, you should never round in the middle of a problem, only at the end—and you should especially never round when working with interest rates.



0-1-32

Imagine a retirement fund that gives 7% compounded weekly. We are going to compute how much someone would have if they deposited \$ 100 per week, every week, for 35 years. We have the periodic rate, i, again. The actual value is

$$i = \frac{0.07}{52} = 0.0013461538461153 \cdots = 0.00134615 \cdots$$

- Albert uses $i = 0.0013$ and his answer is \$ 741,409.93.
- Beatrice uses all available digits, and her answer is \$ 785,145.76.
- Christopher uses $i = 0.00135$ and his answer is \$ 788,920.67.
- We'll also be joined by Dave the drunkard, who uses $i = 0.001$ and his answer is \$ 516,624.83.

As you can see, Christopher is off by \$ 3774.91, which is rather substantial. Even worse, Albert is off by \$ 43,735.83, which could be a really nice used BMW, or perhaps a cheap new one. Dave is actually off by \$ 268,520.93, which can get you a very nice house in Menomonie, Wisconsin.

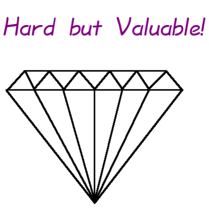
Prior to taking this course, you were probably under the impression that replacing

`0.001346153846154`

on your calculator screen with 0.0013 or 0.00135 was forgivable, or even normal. However, that's simply not true. These errors are large enough that, if made by licensed fiduciaries, could result in going to prison.

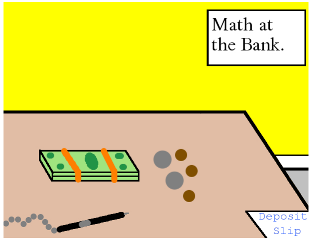
You must never round in the middle of a problem—only at the end.

You have now completed the bulk of this module. The remainder might interest a few highly-motivated readers.



Now for those with die-hard curiosity about the internal mechanics of rounding error, I can explain why, in applied mathematics, we truncate instead of round. Unless you are burning with eagerness to learn about this particular fine point, then you are finished with this module.

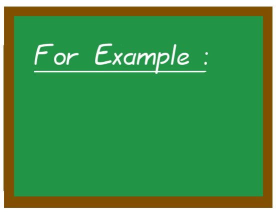
There are two sides: a practical side and a theoretical side.



First, let's discuss the practical side. If the answer should be a real number, whether you truncate or round simply doesn't matter. The first and second significant digits carry the bulk of the importance of a number.

Consider that your company is owed a payment of \$ 123,456,789, but that this has to be negotiated downward. Perhaps the company that owes the payment is having financial trouble. Would you rather see the 8 or 9 get lowered? Or would you prefer to see the 2 or 3 get lowered?

In one case, you might lose \$ 20,000,000 or \$ 3,000,000, but in the other case you'd lose \$ 80 or \$ 9. As you can see, it is the left-end or front-end of the number that is most important. What happens to the right-end or back-end is far, far less important.



0-1-33

Suppose a bunch of students are working on some math homework together, and are all using a different number of significant digits for their final answer. There are students rounding to 2, 3, 4, 5, 6, 7, 8, 9, and 10 significant digits, in fact. Let's see what their answers will look like, if computing an easy problem, such as "What is the square root of 717?"

The calculator would say

$$\sqrt{717} = 26.7768556779917\cdots$$

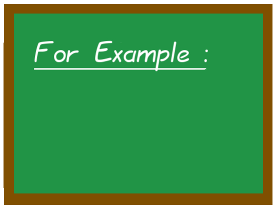
and the following table shows what each student would record, if using truncation or if using rounding.

This is a continuation of the previous box...

Policy	With Rounding	With Truncation
2 significant digits	27.	$26.\cdots$
3 significant digits	26.8	$26.7\cdots$
4 significant digits	26.78	$26.77\cdots$
5 significant digits	26.777	$26.776\cdots$
6 significant digits	26.7769	$26.7768\cdots$
7 significant digits	26.77686	$26.77685\cdots$
8 significant digits	26.776856	$26.776855\cdots$
9 significant digits	26.7768557	$26.7768556\cdots$
10 significant digits	26.77685568	$26.77685567\cdots$

Now look at the right-hand list of numbers. Do you see that each entry is a subset of all the entries below it? Or we could say each is a "prefix" of the entries below it? This is just a wordy way of indicating that all 5 digits of the 5-digit student agree with the first 5 digits of the 7-digit student (and the 8-digit student, the 9-digit student, the 10-digit student, as well as the 6-digit student.)

On the other hand, that's quite false for the left-hand table. That agreement is not present. The digits shown in red ink indicate digits that are "lying." For example, the 4-digit student claims that the number in the hundredth place is an 8, yet six students below him disagree. The entries that disagree in their final entries, the "liar digits," disagree due to rounding.



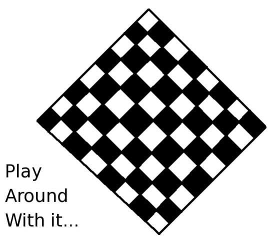
0-1-34

Just to prove to you that the previous example is not some sort of fluke, let's now consider those same students finding $\sqrt{823}$. They'd obtain

Policy	With Rounding	With Truncation
2 significant digits	29.	$28.\cdots$
3 significant digits	28.7	$28.6\cdots$
4 significant digits	28.69	$28.68\cdots$
5 significant digits	28.688	$28.687\cdots$
6 significant digits	28.6880	$28.6879\cdots$
7 significant digits	28.68798	$28.68797\cdots$
8 significant digits	28.687977	$28.687976\cdots$
9 significant digits	28.6879766	$28.6879765\cdots$
10 significant digits	28.68797658	$28.68797657\cdots$

As you can see, in the case of $\sqrt{823}$, just like for $\sqrt{717}$, we see that if we use truncation (just cutting the number off at some point) then what we are really saying is "this number starts with what I have written, and then keeps going with digits I have not written."

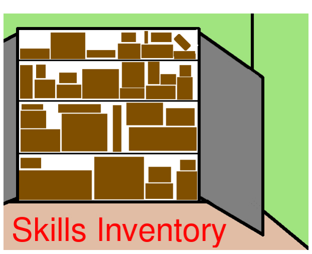
You can check your work in such situations by placing the calculator under the number in question, and seeing if the calculator agrees with the numbers you have written. With truncation, there are no "liar digits" formed by rounding, because we are not rounding.



0-1-35

If you are curious, you could try to produce such a table for $\sqrt{776}$, on your own. (This is easy! Give it a shot!)

The answer is given at the end of the module, on Page 50.]



In this module, you have learned:

- to convert from scientific notation into ordinary notation, both "the long way" and via the shortcut;
- to convert from ordinary notation into scientific notation, both "the long way" and via the shortcut;
- to label the significant digits of a number;
- to state how many significant digits a number is showing;
- to decide what interval of values is equivalent to a given number, when computing with six significant figures.
- We also discussed why I have chosen six significant figures as my standard for almost every problem in this textbook.
- We further discussed "the golden rule of accuracy" which says that you should never round (or truncate) in the middle of a problem—only at the end of a problem.
- We learned the vocabulary terms: least significant digit; most significant digit; rounding; truncation. The vocabulary terms "GDP" and "the G-8 nations" were left for you to look up yourself.

Here are the answers to the question (from Page 32) where we were asked to convert some small numbers from scientific notation into ordinary notation.

- We obtain $5.281 \times 10^{-3} = 0.005281$.
- We obtain $3.491 \times 10^{-4} = 0.0003491$.
- We obtain $6.821 \times 10^{-2} = 0.06821$.
- We obtain $9.807 \times 10^{-1} = 0.9807$.

Here are the answers to the question (from Page 33) where we were asked to convert some small numbers from scientific notation into ordinary notation.

- We obtain $9.724 \times 10^{6} = 9,724,000$.
- We obtain $4.127 \times 10^{8} = 412,700,000$.
- We obtain $176 \times 10^{7} = 1,760,000,000$.
- We obtain $5.68912 \times 10^{4} = 56,891.2$.

Here are the answers to the question (from Page 35) where we were asked to convert some small numbers from scientific notation into ordinary notation.

- We obtain $0.7834 = 7.834 \times 10^{-1}$.
- We obtain $0.002356 = 2.356 \times 10^{-3}$.
- We obtain $0.0123 = 1.23 \times 10^{-2}$.
- We obtain $0.0008754 = 8.754 \times 10^{-4}$.

Here are the answers to the question (from Page 34) about converting "spoken" numbers into scientific notation and then into ordinary notation.

- 4.8 billion dollars $= 4.8 \times 10^{9} = 4,800,000,000$
- 16.5 million dollars $= 1.65 \times 10^{7} = 16,500,000$
- half a billion dollars $= 5 \times 10^{8} = 500,000,000$
- 19.94 quadrillion Japanese Yen $= 1.994 \times 10^{16} = 19,940,000,000,000,000$

Here are the answers to the question (from Page 36) about converting GDPs into scientific notation.

- Canada = 1.319×10^{12}
- France = 2.635×10^{12}
- Germany = 3.235×10^{12}
- Italy = 2.09×10^{12}
- Japan = 5.049×10^{12}
- Russia = 1.232×10^{12}
- The UK = 2.198×10^{12}
- The USA = 1.427×10^{13}

Note: I cannot write 14.27×10^{12} for the USA, because I cannot have a "14" to the left of the decimal point. I can only have a single digit.

Here is the solution to the question (from Page 40) where you were asked how many significant digits a handful of particular given numbers had.

- 13,768,000 has five significant digits.
- 241.589 has six significant digits.
- 0.005600198 has seven significant digits.
- 0.0003125 has four significant digits.
- 17,800,000,000 has three significant digits.

This is the answer to the checkerboard box about $\sqrt{776}$ that was found on Page 48.

Policy	With Rounding	With Truncation
2 significant digits	28.	27.···
3 significant digits	27.9	27.8···
4 significant digits	27.86	27.85···
5 significant digits	27.857	27.856···
6 significant digits	27.8568	27.8567···
7 significant digits	27.85678	27.85677···
8 significant digits	27.856777	27.856776···
9 significant digits	27.8567766	27.8567765···
10 significant digits	27.85677655	27.85677655···