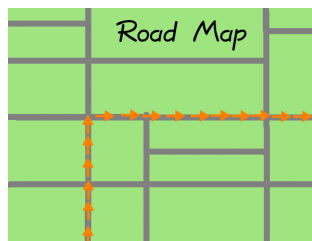## Module 1.2:   Using the Quadratic Formula to Solve Quadratic Equations

Road Map

In this module you will learn how to use the *Quadratic Formula* to find solutions for quadratic equations. The Quadratic Formula is a classic algebraic method that expresses the relationship between a quadratic equation's coefficients and its solutions.

For readers who have already been introduced to the Quadratic Formula in high school, this module will serve as a convenient refresher for the method of applying the formula to quadratic equations. If you are unfamiliar with the formula or have long since forgotten it, you will find it to be an extremely powerful tool for solving quadratic equations, though it does require just a little memorization.

Students who find memorization unpleasant may consider "Completing the Square" as a longer, slower, but memorization-free alternative. Most students find the quadratic formula to be faster, but some students simply cannot memorize the quadratic formula correctly.

This textbook does not have a module for that, yet. However you can find one the website "PurpleMath" at the following URL:
`http://www.purplemath.com/modules/sqrquad.htm`
Note: both methods will always work on all problems of this type.

Valuable Information
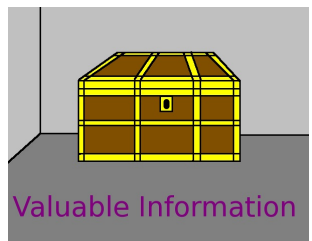
A quadratic equation is an equation that can be written in the form

$$ax^2 + bx + c = 0$$

where a, b, and c are numerical constants, x is a single variable, and $a \neq 0$. In other words, a quadratic equation is a polynomial whose highest-degree term must be raised to the second power. The name "quadratic" comes from the Latin root word "quadratus," meaning "squared." Accordingly, the term that is raised to the second power, ax^2, is commonly called the "squared term."

For Example :

When its terms are listed in decreasing order according to the power of its variable, and 0 is on one side of the equation, as shown above, this is known as the *standard form* of a quadratic equation. To illustrate this point, the following are all valid examples of quadratic equations in standard form:

$$0 = -12x^2 + 1 \qquad 0 = 2z^2 + 9z - 10$$
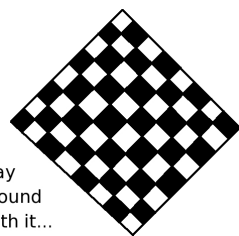$$y^2 = 0 \qquad\qquad -3w^2 - 8w = 0$$
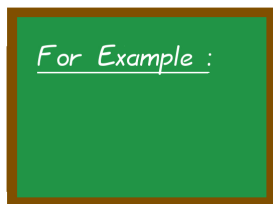
A quadratic equation can come in a variety of forms, the two criteria being that (1) it must include a squared term, and (2) the squared term must be the highest-degree term of the polynomial. Here are some examples of quadratic equations *not* in standard form:

$$2x + x^2 = 1.125 \qquad 0 = -10 + 3x^2 - 4x$$
$$1 + 2x - 7x^2 = 1 \qquad 0.75x = 5 - 30x^2$$

1-2-1

Finally, following are a few examples of equations that are not quadratic at all. Can you tell why?
$$3x^3 + 2x^2 - x - 4 = 0 \qquad 12 = -8x + 4$$
$$7x^4 - 4x^2 + 3 = 0 \qquad x = x^3$$

For each of the following polynomials, identify whether it is (A) not quadratic at all, (B) quadratic but not written in standard form, or (C) quadratic and in standard form:

- $2x^2 - 90x + x^3$ [Answer: A. The equation is not quadratic at all.]
- $6x^2 + 11x - 30$ [Answer: C. The equation is quadratic and in standard form.]
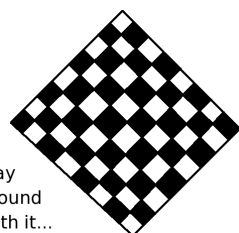- $x^2 - 49$ [Answer: B. The equation is quadratic but not in standard form.]

Play Around With it...

1-2-2

For each of the following polynomials, identify whether it is (A) not quadratic at all, (B) quadratic but not written in standard form, or (C) quadratic and in standard form:

- $3 + 12z + 5z^2$ [Answer: B. The equation is quadratic but not in standard form.]
- $z^4 - z^2$ [Answer: A. The equation is not quadratic at all.]
- $2z^2 + 6z + 28$ [Answer: C. The equation is quadratic and in standard form.]

Play Around With it...

1-2-3

but why?

In this module we will solve many quadratic equations. Our goal in each example will be to calculate the values of x that, when plugged into the equation, yield the result 0. Such values of x are called the *roots*, *zeroes*, or *solutions* to the quadratic equation. Most often a quadratic equation will have two real-valued solutions. You will soon see, however, that some quadratic equations have only one real-valued solution, while others have complex or imaginary solutions; you need not concern yourself with this last group, as we will only be considering real-valued solutions to quadratic equations in this book.

Without further ado,
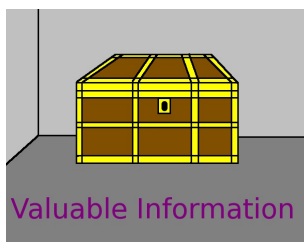
The Quadratic Formula

$$\text{if } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Valuable Information

Suppose you are given a quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are numerical coefficients and $a \neq 0$. Zero *must* be on one side of the equation, and the terms in the polynomial can be in any order. However, descending-power order (also known as standard form), as presented here, is the "cleanest" way to write the expression. To solve this quadratic equation, simply plug your coefficients a, b, and c into the formula and simplify.

Historical Note:

The Quadratic Formula as we know it today—as it was formalized in the previous box—was first published by the philosopher Rene Descartes in his work *La Geometrie* in 1637. However, mankind has been solving quadratic equations as early as four thousand years ago, when the Babylonians handled systems of equations of the form $x + y = p; \ xy = q$, although they found their solutions using strictly geometric rather than algebraic methods.

Later on, Chinese, Indian, and Greek mathematicians also explored solving quadratic equations through geometric reasoning. The Indian mathematician Brahmagupta (597–668 c e) described the solution of quadratic equations through a sequence of algebraic steps that is remarkably similar to the quadratic equation.

By the way, Rene Descartes (1596–1650) also introduced the coordinate plane to mathematics, and in *La Geometrie* a large number of otherwise difficult geometric problems are solved by using coordinates to render them as moderately easy algebra problems. It is for this reason that older math textbooks will refer to the coordinate plane as The Cartesian Plane.

Despite his important mathematical contributions, Descartes is better known for his work in philosophy. Many philosophy textbooks use Descartes to mark the end of the Renaissance and the beginning of the modern era.

I have to restrain myself from telling you the story of Rene Descartes in more detail, because it would take up a lot of space. Of the several books that are biographies of Descartes, I highly recommend *Descartes's Secret Notebook: A True Tale of Mathematics, Mysticism, and the Quest to Understand the Universe*, by Amir Aczel, a short and compact paperback published in 2006.

It includes an excellent coverage of Descartes death, which occurred under suspicious circumstances.

Along with the Pythagorean Theorem and the Compound Interest Formula, the Quadratic Formula is one of those essential formulas that everyone learns eventually and no one should forget.

It will serve you well to know this formula by heart before moving on beyond this module. On the other hand, the exercises in this module are probably enough to cause your brain to absorb the formula (by using the formula), without requiring additional steps to memorize it.

Valuable Information
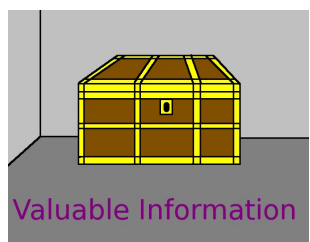
The '$\pm$' in the numerator indicates that if the term under the square root is not zero, then x has two possible values,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Let's see a few examples where this is the case.

Valuable Information
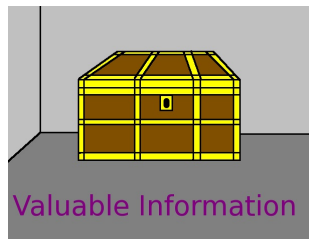
To aid you in your memorization of the formula, there are a number of mnemonic devices available to you, many in the form of a catchy song. The way I learned it—and this is still the primary way I recite it in my head—is with the melody of the children's nursery rhyme "Pop! Goes the Weasel."

A quick internet search will yield countless mnemonic options to choose from, if straightforward rote memorization is not your favorite.

In the equation $x^2 - 3x - 4 = 0$, we can see that $a = 1$, $b = -3$, and $c = -4$. Since the standard form of a quadratic is generalized as $ax^2 + bx + c$, any negative sign that appears must be included in its respective coefficient representation. To find the solutions to this equation, we plug our three coefficients into the Quadratic Formula:

For Example :
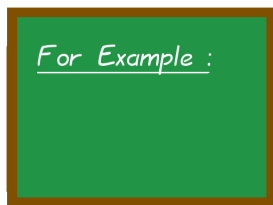
$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{9 - (-16)}}{2}$$

$$x = \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2}$$

\# 1-2-4

This formula gives us the zeroes

$$x = \frac{3 + 5}{2} = \frac{8}{2} = 4, \text{ and } x = \frac{3 - 5}{2} = \frac{-2}{2} = -1$$

leading to the solutions $x = 4$ or $x = -1$. (When two solutions are possible "in x," we express the solutions as "$x = a$ and/or $x = b$.")

Above, the equation we were trying to solve was $x^2 - 3x - 4 = 0$. We expect that by plugging our two possible solutions into the equation—one at a time—we will obtain the identity $0 = 0$ for each:

Check Your Work !!

$$x = 4 \qquad\qquad x = -1$$
$$(4)^2 - 3(4) - 4 \overset{?}{=} 0 \qquad (-1)^2 - 3(-1) - 4 \overset{?}{=} 0$$
$$16 - 12 - 4 \overset{?}{=} 0 \qquad 1 - (-3) - 4 \overset{?}{=} 0$$
$$0 = 0 \qquad\qquad 0 = 0$$
$$(\text{yes!}) \qquad\qquad (\text{yes!})$$

Having arrived at the identity $0 = 0$ after plugging in both values of x, we can confirm that the two solutions to the quadratic equation $x^2 - 3x - 4 = 0$ are $x = 4$ or $x = -1$.

I would like to stress the importance of taking the time to routinely check your work, especially when solving quadratic equations. Checking your work takes only a moment, and it can really improve a quiz or exam score.
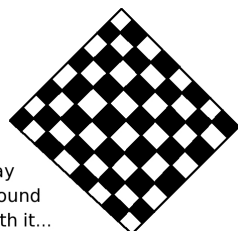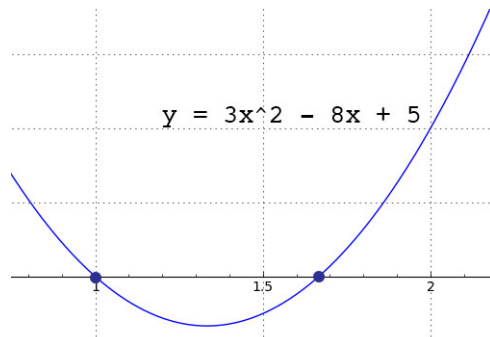
Find the roots for the following quadratic equations. Each equation will have two unique solutions.

Play Around With it...

- Solve $x^2 - 5x - 14 = 0$ \hfill [Answer: $x = -2$ or $x = 7$.]

- Solve $3x^2 - 8x + 5 = 0$ \hfill [Answer: $x = 1$ or $x = \frac{5}{3}$.]

\# 1-2-5

$$y = 3x^2 - 8x + 5$$

There is a deliberate reason why I have been alternately describing the solutions of quadratic equations as "roots" and "zeroes." When graphed on a coordinate plate, the quadratic function $y = ax^2 + bx + c$ traces a parabola, which is a curve that resembles a bowl facing either up or down (think para-"bowl"-a). The values of x we are curious about are those that describe where the parabolic curve intersects the x-axis, or the line $y = 0$. Thus these x-values are called the *roots* or *zeroes* of the quadratic function.

DANGER !!!

Remember this important rule when solving quadratic equations: before you determine the constant values of a, b, and c, it is imperative that you move all terms in the equation to one side of the equal sign, leaving 0 on the other side.

Consider the quadratic $x^2 - 5x + 6 = 6$. As it comes to pass, if you were to (incorrectly) use the Quadratic Formula with $a = 1$, $b = -5$, and $c = 6$, then you would obtain the (incorrect) result $x = 2$ or $x = 3$. (Try it yourself, if you'd like.) However, if you plug these in as your potential solutions, then you will notice that they do not satisfy the given equation. In other words, when $x = 2$ or $x = 3$, we get $x^2 - 5x + 6 = 0 \neq 6$, and we need the result from the left-hand side to be 6.

Instead, you have to move the 6 on the right-hand side of the original equation over, to get $x^2 - 5x + 0 = 0$, and then proceed with plugging $a = 1$, $b = -5$, and $c = 0$ into the Quadratic Formula. At that point you will get $x = 0$ and $x = 5$ as the two true solutions to this quadratic. (Feel free to try this yourself.) Indeed, when we plug these results into the original $x^2 - 5x + 6 = 6$, we find that the equation is satisfied. (Try this yourself as well, as a check of your work.)

Therefore, it is crucial that you always move all terms in a quadratic equation over to one side, setting the polynomial equal to zero, before using the Quadratic Formula.

For Example :

1-2-6

Sometimes the solutions to a quadratic equation will contain irrational numbers. Here is such an example. Solve $2x^2 - 12x + 11 = 0$.

Observing that $a = 2$, $b = -12$, and $c = 11$,
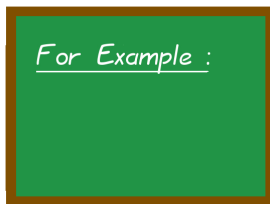
$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(2)(11)}}{2(2)}$$

$$x = \frac{12 \pm \sqrt{144 - 88}}{4}$$

$$x = \frac{12 \pm \sqrt{56}}{4}$$

$$x = \frac{12 + \sqrt{56}}{4} \qquad x = \frac{12 - \sqrt{56}}{4}$$

$$x = 4.87082\cdots \qquad x = 1.12917\cdots$$

This equation has two unique solutions, $x = 4.87082\cdots$ and $x = 1.12917\cdots$.

Check Your Work !!

I encourage you to check these two possible solutions by plugging them into the original equation. The easiest way to do this would be to plug in an x-value, store that value in a variable on your calculator, and then plug the equation into your calculator using that variable. Six decimal places of precision should do the trick to yield an expression that is close enough to 0 to reassure you. Once in a great while you might get a response that is close to zero, but not close enough to make you confident. In this case, go ahead and check the work again with all available digits.

Math at the Bank.

You might be accustomed from your algebra coursework to write answers in a style not unlike

$$x = \frac{-1 + \sqrt{5}}{2}$$
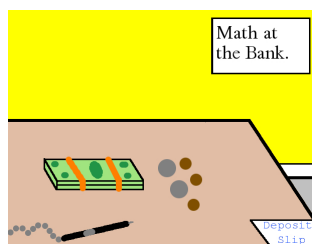
but we will not do that in this textbook. The reason is that decimal real numbers, sometimes called floating-point numbers, are what would be used in any economic, scientific, or financial situation. In this case, the above value should be expressed as

$$x = 0.618033\cdots$$

as we are computing to six significant digits throughout this textbook.

Consider it this way... If you asked for $\sqrt{23}$ pounds of sugar, or $\sqrt{7654321}$ Yen, how would this be interpreted? If you request a block of wood $\sqrt{29}$ inches long, or a certificate of deposit with interest rate $\sqrt{1.1}$, then assuming the other person doesn't laugh you out of their office, they would comply with your request by converting what you asked for into a decimal real number. Of course, in reality, you should do that conversion yourself.

Play Around With it...
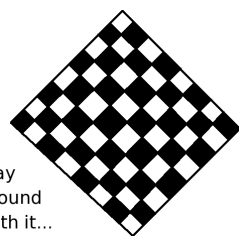
1-2-7

Calculate the solutions to the following equations:

- Solve $-2q^2 + 10q + 12 = 0$ [Answer: $q = -1$ or $q = 6$.]
- Solve $x^2 + x - 1 = 0$ [Answer: $x = 0.618033\cdots$ or $x = -1.61803\cdots$.]
- Solve $4s^2 - 2s = 30$ [Answer: $s = 3$ or $s = -2.5$.]
- Solve $-3z - 5z^2 = -9$ [Answer: $z = -1.67477\cdots$ or $z = 1.07477\cdots$.]

For Example :
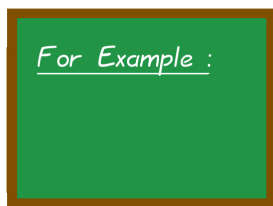
1-2-8

Earlier you read that the only requirement for a quadratic equation is that its highest-degree term is squared. We will explore two examples in which either the b- or c-value is 0 when considering the equation in standard form.

Solve $3x^2 - 20 = 0$. In this example, $a = 3$, $b = 0$ (because there is no first-degree x term in the equation), and $c = -20$. Plugging into the Quadratic Formula,

$$x = \frac{-(0) \pm \sqrt{(0)^2 - 4(3)(-20)}}{2(3)}$$

$$x = \frac{\pm\sqrt{240}}{6}$$

$$x = \frac{+\sqrt{240}}{6} \qquad x = \frac{-\sqrt{240}}{6}$$

$$x = 2.58198\cdots \qquad x = -2.58198\cdots$$

We arrive at the solutions $x = 2.58198\cdots$ and $x = -2.58198\cdots$.
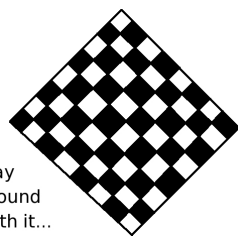
Find the zeroes for the quadratic equations.

Play
Around
With it...

1-2-9

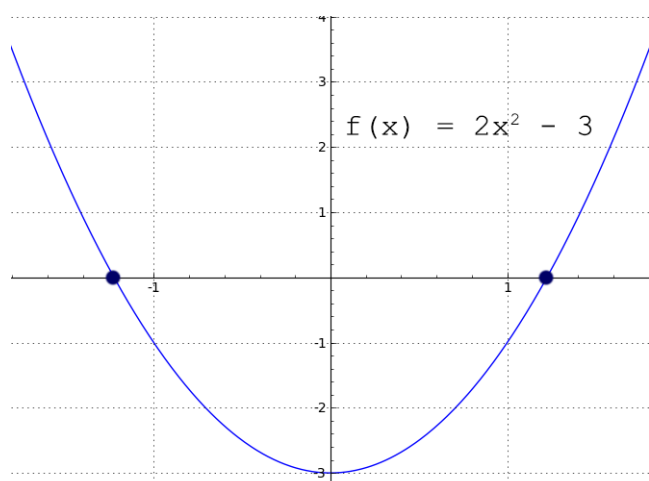- Solve $2x^2 - 3 = 0$ [Answer: $x = 1.22474 \cdots$ or $x = -1.22474 \cdots$.]

- Solve $40 - 5y^2 = 0$ [Answer: $y = 2.82842 \cdots$ or $y = -2.82842 \cdots$.]

- Solve $w^2 - 48 = 0$ [Answer: $w = 6.92820 \cdots$ or $w = -6.92820 \cdots$.]


$f(x) = 2x^2 - 3$

Perhaps you noticed this pattern emerging in the previous two boxes, but quadratic equations of the form $ax^2 + c = 0$ (where there is no first-degree x term) have very predictable solutions.

You will easily observe that for a quadratic where $b = 0$, the solutions are always *additive inverses* of one another (they sum to zero). If s is a solution, then so is $-s$. This is not an accident: equations of the form $y = ax^2 + c$ represent a parabolic curve symmetric along the y-axis, shifted upward (vertically) by c units. An upward shift does not destroy the symmetry of the curve, so the curve's intersection with the positive region of the x-axis is the same distance from the origin as the curve's intersection with the negative region of the x-axis.

Now we will see an example in which the c-value is 0. Find the roots of the equation $-x^2 + 5x = 0$.

Noting that $a = -1$, $b = 5$, and $c = 0$,
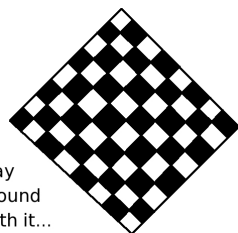
For Example :
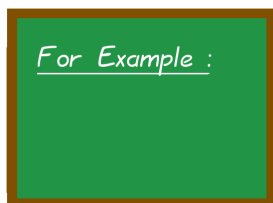
$$x = \frac{-(5) \pm \sqrt{(5)^2 - 4(-1)(0)}}{2(-1)}$$

$$x = \frac{-5 \pm \sqrt{25}}{-2}$$

$$x = \frac{-5 + 5}{-2} \qquad x = \frac{-5 - 5}{-2}$$

$$x = 0 \qquad x = 5$$

1-2-10

Our formula gives us the zeroes $x = 0, 5$.

Find the roots for the quadratic equations.

Play
Around
With it...

1-2-11

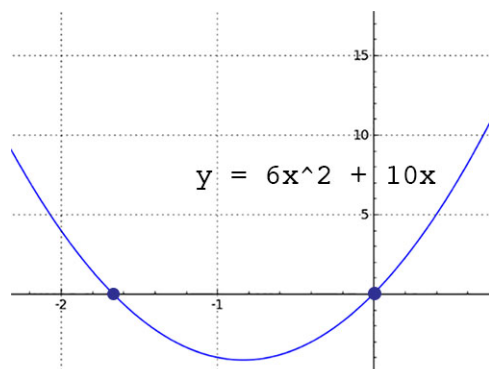- Solve $-4x^2 + 17x = 0$ [Answer: $x = 0$ or $x = \frac{17}{4}$.]

- Solve $10y + 6y^2 = 0$ [Answer: $y = -\frac{5}{3}$ or $y = 0$.]

- Solve $3z^2 + 8z = 0$ [Answer: $z = 0$ or $z = -\frac{8}{3}$.]

When solving quadratics of the form $ax^2 + bx + c = 0$, a pattern also emerges among the solutions when $c = 0$, as is illustrated in the plot to the left. First, for equations of this variety, one real-valued solution will always be at the origin, or $x = 0$. The second real-valued solution will always be $x = \frac{-b}{a}$ (assuming a and b are real, of course). Note that, in this scenario it is implicit that $a \neq 0$, since we are dealing with strictly quadratic equations, which must have a non-zero coefficient in front of their squared term.

To the left, the line $y = 6x^2 + 10x$ intersects the x-axis at $x = 0$ and $x = -\frac{5}{3}$. I encourage you to review the previous two boxes and check for yourself that the solutions to those equations follow the format I have just described.

y = 6x^2 + 10x
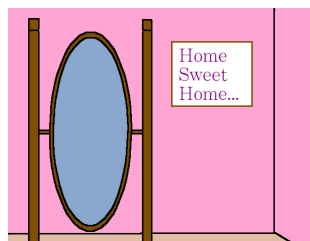
A Pause for Reflection. . .

So far, for each quadratic equation we've seen there have been exactly two real-valued solutions. It turns out that two other outcomes are possible when solving quadratics: there may be just one single solution, or there may be two solutions whose values are complex (which we will treat as "error" values in this textbook). Fortunately, there is a method to determine—even before implementing the Quadratic Formula—both the number and type of solutions you should expect for a particular quadratic equation. This method is based on the analysis of something called the *discriminant*.

Home Sweet Home...

In the Quadratic Formula, the term underneath the square root $(b^2 - 4ac)$ is known as the *discriminant*, which will be stylized as D in this book. Its value, when plugging in for a, b, and c, determines what kind of solutions a quadratic equation has. When comparing the discriminant to the real number 0,

$$\text{if} \quad b^2 - 4ac > 0 \quad \implies \quad \text{2 distinct real solutions}$$

$$\text{if} \quad b^2 - 4ac = 0 \quad \implies \quad \text{1 distinct real solution, } x = -\frac{b}{2a}$$

$$\text{if} \quad b^2 - 4ac < 0 \quad \implies \quad \text{2 distinct complex or imaginary (non-real) solutions}$$
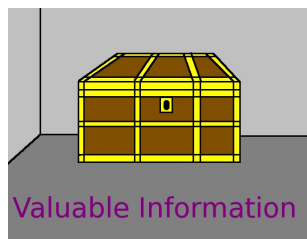
Without diving into topics that are beyond the scope of this book, all you must know about complex numbers is that they contain a term which is a multiple of $\sqrt{-1}$, an expression that exists outside the real numbers because one cannot find a square root of a negative real number. One example of a complex number is $(4 + \sqrt{-2})$.

If you are solving for a quadratic equation and you find that the expression underneath the square root sign—what we now call the *discriminant*—evaluates to a negative number, consider this an error and simply report "the equation has no real solutions" or "the roots are complex/imaginary." We are not concerned with complex or imaginary values in this book.

Valuable Information

Some books will write the Greek letter "Δ," called "delta," for the discriminant, but the symbol Δ in finance and economics usually represents change of some sort. We will see Δ used that way frequently later in this book, so we will use D for the discriminant when we need it.

For Example :

1-2-12

Find the zeroes of $-2x^2 + 8x - 8 = 0$.

Here we have $a = -2$, $b = 8$, and $c = -8$. The discriminant, $b^2 - 4ac$, evaluates to

$$(8)^2 - 4(-2)(-8) = 64 - 64 = 0$$

so we expect there to be only a single real solution to this equation. Let's verify this by plugging into the Quadratic Formula, as we are accustomed to doing by now:
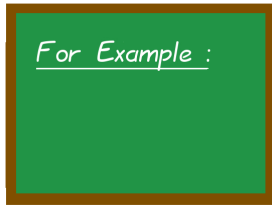
$$x = \frac{-8 \pm \sqrt{0}}{2(-2)} \quad \text{(we already evaluated } b^2 - 4ac\text{)}$$

$$x = \frac{-8 \pm 0}{2(-2)}$$

$$x = \frac{-8}{-4}$$

$$x = 2$$

We find, as expected, that the equation only has one real-valued solution, $x = 2$.

However, you must not forget to plug this value back into the equation to check your work.

Check Your Work !!

Plug $x = 2$ into the quadratic equation from the previous box to check that it is in fact a solution.

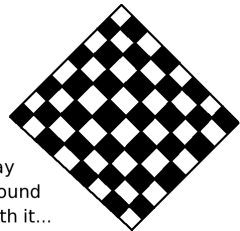$$-2(2)^2 + 8(2) - 8 \overset{?}{=} 0$$

$$-2(4) + 16 - 8 \overset{?}{=} 0$$

$$-8 + 8 \overset{?}{=} 0$$

$$0 = 0$$

$$\text{(yes!)}$$

Play Around With it...

1-2-13

Find the roots to each quadratic equation:

- $\left(\dfrac{7x}{4}\right)^2 - 7x + 4 = 0$ [Answer: $x = \dfrac{8}{7}$. (The discriminant equals zero.)]

- $-9w^2 + 12w - 4 = 0$ [Answer: $w = \dfrac{2}{3}$. (The discriminant equals zero.)]

- If you did not quite get the first solution right, did you notice that $a = \frac{49}{16}$?

For Example :

1-2-14

Find the real solutions to the equation $x^2 - x + 1$.
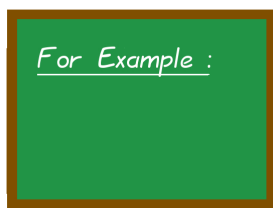
$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1 - 4}}{2}$$

$$x = \frac{1 \pm \sqrt{-3}}{2}$$

We see that either possible solution will contain the term $\sqrt{-3}$, which is not a real number since a real square root of -3 does not exist. Therefore, we conclude that no real solutions exist for this quadratic.


Home Sweet Home...

A Pause for Reflection. . .
We should have anticipated this result, actually. If we had simply plugged our coefficients into the discriminant formula ($b^2 - 4ac$) beforehand, we would have seen that it evaluates to -3, which is less than 0, and could have concluded that no real solutions exist. In this case, finding the discriminant first would have saved us some calculation steps.

In fact, I am told that in one Eastern European country (I have forgotten which), they do not write the quadratic equation as we do. Students there are taught to first compute the discriminant D, and then compute
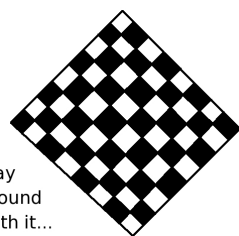
$$x = \frac{-b \pm \sqrt{D}}{2a}$$

which in many cases is a timesaver. This method also makes it easier for the instructor to give partial credit.

Play Around With it...
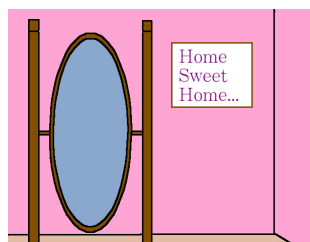
1-2-15

Do the following quadratic equations have real-valued solutions?

- $5x - 3 - 3x^2 = 0$ [Answer: No. The discriminant equals -11, which is less than zero.]
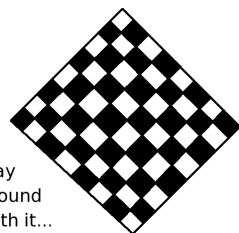- $2x^2 + 4 + 6x = 0$ [Answer: Yes. $b^2 - 4ac = 4$, which is greater than zero.]

Play Around With it...

1-2-16

In each of the following exercises, first determine the value of the discriminant D, then find all real solutions:

- Solve $x^2 - 10x + 25 = 0$ [Answer: $D = 0$, and $x = 5$.]
- Solve $7z^2 + 3z + 4 = 0$ [Answer: $D = -103 < 0$, thus no real solutions exist.]
- Solve $2x^2 + 5x + 3 = 0$ [Answer: $D = 1 > 0$, and $x = -\frac{3}{2}, -1$.]

Find the roots for the following quadratic equation: $-4x^2 + 6 = 3x - 3$.

    First, you must get the equation into standard form by setting one side equal to zero. Here I will move all terms to the left side. The equation can be rewritten as $-4x^2 - 3x + 9 = 0$, where $a = -4$, $b = -3$, and $c = 9$.

For Example :

$$x = \frac{-(-3) \pm \sqrt{9 + 144}}{-8}$$

$$x = \frac{3 \pm 3\sqrt{17}}{-8}$$

$$x = \frac{3 - 3\sqrt{17}}{8} \qquad x = \frac{3 + 3\sqrt{17}}{8}$$

$$x = -1.17116\cdots \qquad x = 1.92116\cdots$$

1-2-17

Because these solutions are rather awkward, I recommend checking their validity yourself.

In the previous box, when we consolidated all of the terms onto the left side of the equation, we wound up with the expression $-4x^2 - 3x + 9 = 0$. Would the solutions to the quadratic change if, instead, we moved all of the terms to the right side (effectively multiplying the expression by -1)?

    Let's work it out for ourselves:

$$\begin{aligned} -4x^2 + 6 &= 3x - 3 \\ 0 &= 3x - 3 + 4x^2 - 6 \\ 0 &= 4x^2 + 3x - 9 \end{aligned}$$

For Example :

As you can see, the coefficients become $a = 4$, $b = 3$, and $c = -9$. Plugging these into the Quadratic Formula, we get

$$x = \frac{-3 \pm \sqrt{3^2 - 4(4)(-9)}}{2(4)} = \frac{-3 \pm \sqrt{9 + 144}}{8}$$

$$x = \frac{-3 \pm 3\sqrt{17}}{8}$$

$$x = -1.92116\cdots \qquad x = 1.17116\cdots$$

1-2-18

These are the same solutions we obtained in the previous box. It turns out that, in general, you can move the terms of a quadratic equation to either side of the equal sign and the roots will remain the same. Therefore, how you simplify a quadratic equation before solving is simply a matter of preference and will not affect your answers.
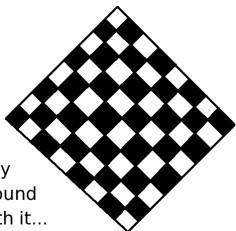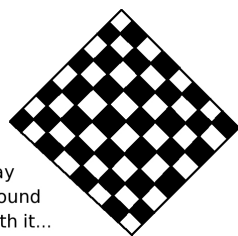

Find the roots for the following quadratic equations:

- Solve $3x^2 + 10x + 10 = x^2 + 3x + 25$ [Answer: $x = 1.5$ or $x = -5$.]

- Solve $5 + 4x - 7x^2 = 10x - 6$ [Answer: $x = -1.75337\cdots$ or $x = 0.896231\cdots$.]

- Solve $5x^2 - 8x + 2 = 7x^2 + 2x - 13$ [Answer: $x = -6.20809\cdots$ or $x = 1.20809$.]

- Solve $4x^2 + 2 = 8x^2 - 14x - 11$ [Answer: $x = -0.762468\cdots$ or $x = 4.26246\cdots$.]

Play Around With it...

# 1-2-19

Play
Around
With it...

1-2-20

The quadratic equation can work well with decimals. This problem, however, will demonstrate how sensitive the quadratic equation is to rounding error. You will want to use nine significant figures for this one. Solve

$$0.01x^2 + 85x + 0.02 = 0$$

[Answer: $x = -0.000235294124\cdots$ and $x = -8499.99976\cdots$.]
  Note, your answer should match mine to eight significant figures, or at least seven.
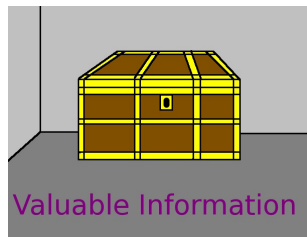
Valuable Information

In a quadratic equation, you can reverse the sign of every term and still get the same solutions. This is equivalent to multiplying the equation by -1. In fact, you can multiply a quadratic equation by *any* real number (whole, decimal, even irrational!) and obtain the same solutions.

  This trick is especially useful when one or all of the coefficients are fractions, and you wish to express the equation strictly in terms of whole numbers. In this case, you would multiply both sides of the equation by the least common denominator of the fractional coefficients. The 0 on one side will obviously be unaffected, but you will now have the convenience of working with a-, b-, and c-values that are whole numbers. This feature will make simplifying the Quadratic Formula much easier than otherwise working with squares and square roots of decimal values.

In the following two boxes we will confront a quadratic equation with fractional coefficients. To solve, do we first scale the quadratic—eliminating fractions entirely—so that we can work with integer coefficients? Or should we just plug fractions into the Quadratic Formula? I will now show you evidence that indicates the former method is preferred.
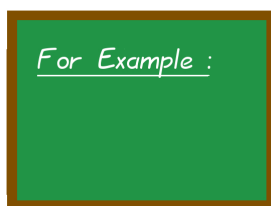
For Example :

1-2-21

Suppose we are asked to solve
$$\frac{1}{7}x^2 - \frac{10}{7}x + 3 = 0$$

  Let's first eliminate fractional coefficients by multiplying the equation by the common denominator 7.

$$7 \cdot \left(\frac{1}{7}x^2 - \frac{10}{7}x + 3 \right) \qquad = (7)(0) = 0$$
$$x^2 - 10x + 21 \qquad = 0$$

At this point, we can employ the Quadratic Formula to find real-valued roots of $x^2 - 10x + 21 = 0$.

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(21)}}{2(1)}$$
$$x = \frac{10 \pm \sqrt{100 - 84}}{2}$$
$$x = \frac{10 + 4}{2} \qquad x = \frac{10 - 4}{2}$$
$$x = 7 \qquad x = 3$$

So, $x = 7$ or $x = 3$. That was relatively painless. In the next box we will see how difficult this problem becomes when we don't eliminate fractions in the beginning.

Now we will solve $\frac{1}{7}x^2 - \frac{10}{7}x + 3 = 0$ by plugging the fractional coefficients into the Quadratic Formula, without first scaling the equation. Notice how messy the work becomes.

$$x = \frac{-\left(-\frac{10}{7}\right) \pm \sqrt{\left(-\frac{10}{7}\right)^2 - 4\left(\frac{1}{7}\right)3}}{2\left(\frac{1}{7}\right)}$$

$$= \frac{\frac{10}{7} \pm \sqrt{\frac{100}{49} - \frac{12}{7}}}{\frac{2}{7}}$$

$$= \frac{\frac{10}{7} \pm \sqrt{\frac{100}{49} - \frac{84}{49}}}{\frac{2}{7}}$$

$$= \frac{\frac{10}{7} \pm \sqrt{\frac{16}{49}}}{\frac{2}{7}} \qquad \leftarrow \text{Never do this! It is hideous!!}$$

$$= \frac{\frac{10}{7} \pm \frac{4}{7}}{\frac{2}{7}}$$

$$= \frac{\frac{14}{7}}{\frac{2}{7}} \text{ or } \frac{\frac{6}{7}}{\frac{2}{7}}$$

$$= \frac{14}{7} \times \frac{7}{2} \text{ or } \frac{6}{7} \times \frac{7}{2}$$

$$x = \frac{14}{2} \text{ or } \frac{6}{2} = 7 \text{ or } 3$$

DANGER !!!

Although we cannot call this the "wrong way" to solve a quadratic, it certainly is the "long way." The moral is, whenever possible, eliminate fractional coefficients before employing the Quadratic Formula. This will save you the headache of dividing fractions by fractions.
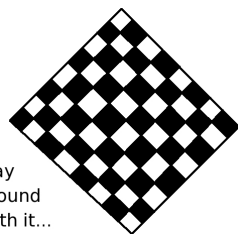
Play Around With it...

Find the zeroes for the following equations:

- Solve $2z^2 - z = 0.5z + 5$ [Answer: $z = -\frac{5}{4}$ or $z = 2$.]
- Solve $\frac{1}{3}x^2 - \frac{1}{4}x - 3 = \frac{5}{6}(x+1)$ [Answer: $x = -2.13540\cdots$ or $x = 5.38540\cdots$.]
- Solve $-\frac{3}{4} + 5x - 0.6x^2 = \frac{5}{2}x - 6$ [Answer: $x = -1.53471\cdots$ or $x = 5.70138\cdots$.]

\# 1-2-22

For Example :

While the Quadratic Formula will always produce for you the real solutions to a quadratic equation (if they exist), do not feel compelled to use this formula if you can "spot" the roots of an equation an easier way. The Quadratic Formula is just one method that guarantees an answer, but it is not the only method.

For example, if you are given the equation $x^2 - 4 = 0$, you might recognize that the left-hand expression is a difference of squares and be able to factor it into $(x-2)(x+2) = 0$ automatically, without having to resort to the formula.

\# 1-2-23
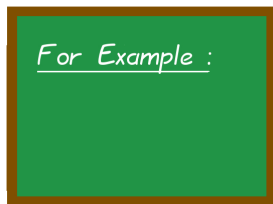
For Example :

1-2-24

Likewise, an equation for which the c-value is 0 lends itself to simply factoring out an x: $ax^2 + bx = 0$ can be reduced quickly to $x(ax + b) = 0$, which leads to the solutions $x = 0$ or $x = -\frac{b}{a}$.

Consider, for example, the equation $10x^2 + 3x = 18x$. In standard form this becomes $10x^2 - 15x = 0$. By factoring out an x from the left-hand side, we get $x(10x - 15) = 0$. If we were to plug 0 in for x, then the left-hand side would become 0, so $x = 0$ is one solution.

The other solution is found by considering the polynomial $(10x - 15)$—if this polynomial evaluates to 0 with some value of x, then the left-hand side of the quadratic will also become 0. Luckily, $10x - 15 = 0$ is a single-variable linear equation, which we can solve easily. Solving for this equation,

$$10x - 15 = 0$$
$$10x = 15$$
$$x = \frac{15}{10} = \frac{3}{2}$$

The second solution we find, $\frac{3}{2} = x$, is equal to $-\frac{(-15)}{10}$, which is exactly the ratio $-\frac{b}{a}$ using coefficients from the original quadratic.

Valuable Information
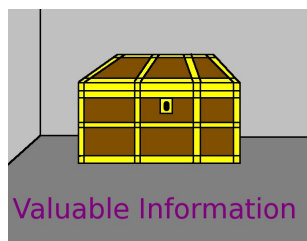
The point is, although the Quadratic Formula will always work to find you existing solutions to a quadratic equation, it can sometimes be computational overkill when more direct methods present themselves to you; this largely depends upon the setup of the equation you are solving for. Nevertheless, when in doubt, use the formula.

Play Around With it...
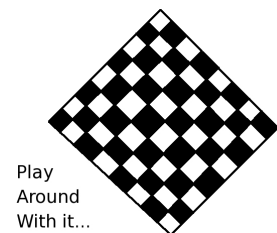
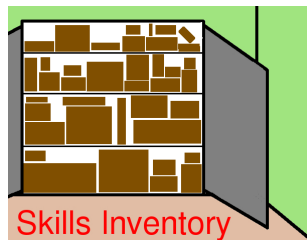# 1-2-25

The following four quadratic equations have either $b = 0$ or $c = 0$. Find their zeroes.

- Solve $3x - 7x^2 = -10x^2 + 12x$ [Answer: $x = 0$ or $x = 3$.]
- Solve $-12x^2 + 15 = 10$ [Answer: $x = -0.645497\cdots$ or $x = +0.645497\cdots$.]
- Solve $x^2 - 6 = 9 - 8x^2$ [Answer: $x = +1.29099\cdots$ or $x = -1.29099\cdots$.]
- Solve $14x^2 + 11x - 6x^2 = 8x - 5x^2$ [Answer: $x = 0$ or $x = -\frac{3}{13}$.]

Skills Inventory

In this module, we learned or reviewed:

- the Quadratic Formula, and how it is used to find real roots (a.k.a. "zeroes" or "solutions") of quadratic equations.

- what a *discriminant* is, and how its value relative to 0 reveals the number of real solutions a quadratic equation has.

- the possible number of real solutions to a quadratic equation: either two, one, or no real solutions, depending on the value of the discriminant.

- patterns in the solutions when either $b = 0$ or $c = 0$ in the equation $ax^2 + bx + c = 0$

- quicker alternatives to the Quadratic Formula, when they present themselves.

- the vocabulary terms *standard form*, *root*, *zero*, and *discriminant*.