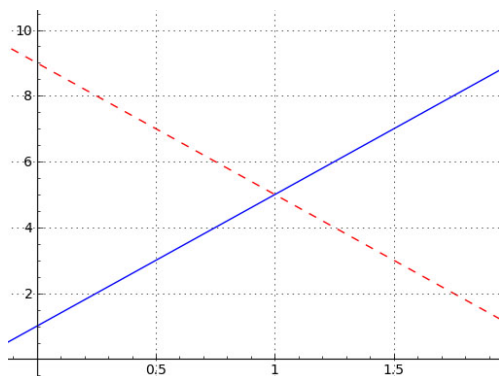


Module 1.4: Intersecting Two Lines, Part One



This module will explain to you several common methods used for intersecting two lines. By this, we mean finding the point (x, y) at which two lines cross one another in the coordinate plane.

Lines are actually graphical representations of linear equations in two variables—often written in the form $y = mx + b$ —so what you will really be learning is how to solve *systems of two linear equations* by computing the (x, y) values that make both equations simultaneously true.



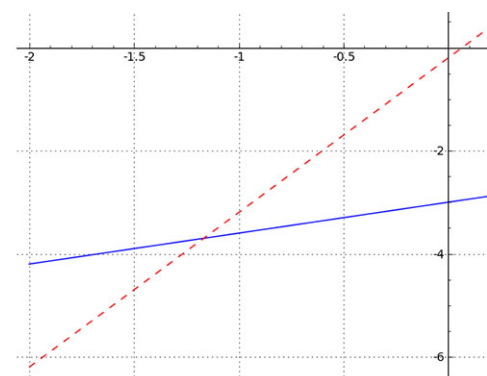
In the coordinate plane shown here, I have plotted two linear equations of the form $y = mx + b$, where m and b are constants. The solid blue line maps the equation $y_1 = 4x + 1$, while the dashed red line maps the equation $y_2 = -4x + 9$.

If I were curious about where these two lines intersect on the plane, I could find the point at which they cross, which appears to lie directly on the gridline for $x = 1$, and exactly between the gridlines $y = 4$ and $y = 6$. I would speculate that the intersection between the two lines occurs at the solitary coordinate $(1, 5)$.

To test my possible solution, I plug $x = 1$ into the linear equations, and hope to arrive at $y = 5$ for both.

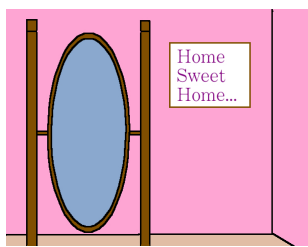
$$\begin{array}{rclcl} y_1 & = & 4(1) + 1 & & y_2 = -4(1) + 9 \\ y_1 & = & 4 + 1 & & y_2 = -4 + 9 \\ y_1 & = & 5 & & y_2 = 5 \\ & & \text{(yes!)} & & \text{(yes!)} \end{array}$$

This result tells me that at $x = 1$, both lines pass through the line $y = 5$. Therefore, the two lines intersect one another at—and only at—the coordinate $(1, 5)$.



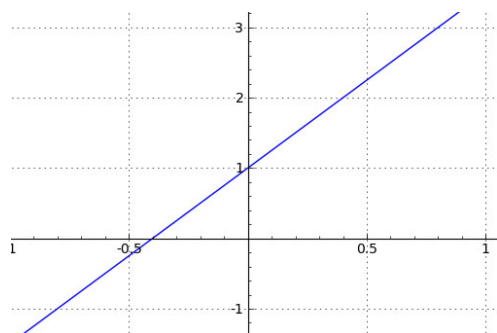
Examining a graph, as we just did, is certainly useful when one wants to approximate the intersection of two lines. However, this method is not reliable for finding exact solutions (which is what we desire) when the linear equations intersect at decimal-valued coordinates.

For example, the lines $y_1 = 0.6x - 3$ and $y_2 = 3x - 0.2$ intersect at the (x, y) coordinate $(-\frac{7}{6}, -3.7)$. This is not a point that one can derive by simply “eyeballing” the graph, especially when gridlines are sparse or nonexistent. And “guessing-and-checking” various approximations to the true (x, y) solution is neither an efficient nor useful expenditure of your time, so it will be strictly avoided. Rather, this module will cover purely algebraic methods for intersecting two lines, which guarantee exact solutions every time (if solutions exist). Nonetheless, an occasional graph can help with visualizing a particular situation.



A Pause for Reflection...

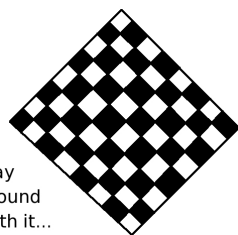
If I include a graph in the remainder of this module, it is only to illustrate some mathematical theory related to the behavior between lines on a plane. Any solutions we obtain throughout this module will be the result of computation with reliable algebraic methods. We will utilize the unbiased rules of algebra, and not rely on our fallible eyesight, so we can be certain that our solutions to the following examples will be 100% accurate the first time around.



A single line on the plane is a graphical representation of the linear equation $y = mx + b$, where m is the slope (rate of change) of the line and b is the y -value at which the line intersects the y -axis. The line graphed to the left is $y = 2.5x + 1$.

Any “solution” to a two-variable linear equation is an (x, y) coordinate pair that, when plugged into the original equation, yields a true mathematical expression. For example, an obvious solution point is the coordinate $(0, 1)$: if you plug $x = 0$ and $y = 1$ into the equation, you obtain the true statement $1 = 1$.

Can you tell whether a coordinate pair lies on a particular line? For each bullet, state whether the given (x, y) pair is a solution to the corresponding linear equation. Note the various ways to express your answer.



Play
Around
With it...

1-4-1

- Is $(\frac{3}{2}, 8.5)$ a solution to $y = 7x - 2$?

[Answer: Yes, that pair is a solution to the linear equation.]

- Does $(45, 750)$ lie on $y = -20x + 180$?

[Answer: No, that point does not lie on the line.]

- Does the line $y = 5x + 20$ contain the point $(-4, 0)$?

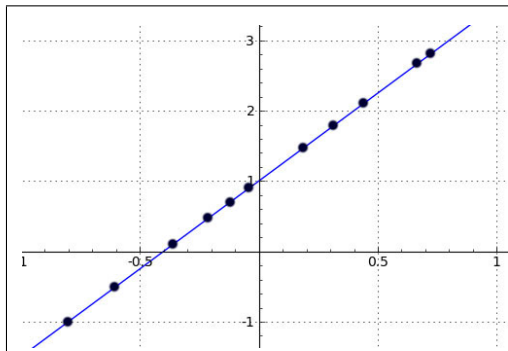
[Answer: Yes, the line contains that point.]

- Is the coordinate $(-4, -\frac{8}{3})$ on the line $y = -\frac{5}{3}x - 9$?

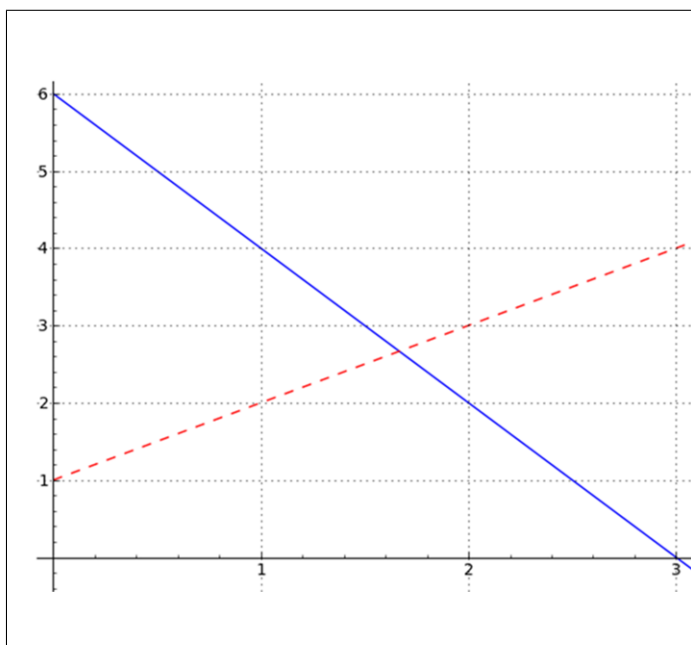
[Answer: No, that coordinate is outside the line.]



It is misleading, however, to ask for “a solution” to a single linear equation in two variables. A line is composed of an infinite number of points, thus there are an infinite number of solutions. Furthermore, the equation for the line is itself a compact expression of the complete collection of solutions—for any x -value you plug in, you can find its corresponding y -value on the line, and thus have a solution point.



To illustrate the above ‘Danger!’ box, I have plotted the line $y = 2.5x + 1$ and highlighted a number of individual points that lie on the line. Any of these coordinate points can be plugged into the linear equation to produce a mathematical truth. In fact, every point on this line—of which there are infinitely many—is a solution to the linear equation. This is the case for any single equation of the form $y = mx + b$.



A more interesting endeavor is discovering the solution to a *system of linear equations*.

A system of linear equations is a collection of two or more linear equations which you must simultaneously solve. It can be represented as two or more lines on the coordinate plane. Except for two rare situations (which we will discuss very shortly), a pair of lines will intersect at exactly one coordinate point. Plugging this pair of (x, y) -values into either of the individual equations will yield a true mathematical expression.

The graph on the left plots the linear system

$$\begin{cases} y = x + 1 \\ y = -2x + 6 \end{cases}$$

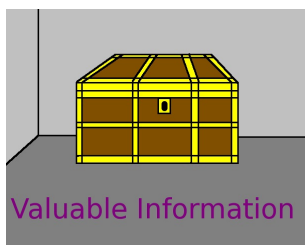
The lines intersect—or, the system is solved—at the (x, y) coordinate $(\frac{5}{3}, \frac{8}{3})$. We will check this solution in the next box. In a couple of pages, you will learn how to determine the solution to a linear system yourself.

You should check that the point $(\frac{5}{3}, \frac{8}{3})$ does in fact represent the solution to the above linear system. To do this, plug the x - and y -values into both equations and see that a true expression is produced from both.



$$\begin{array}{ll} y = x + 1 & y = -2x + 6 \\ \frac{8}{3} = \frac{5}{3} + 1 & \frac{8}{3} = -2\left(\frac{5}{3}\right) + 6 \\ \frac{8}{3} = \frac{5}{3} + \frac{3}{3} & \frac{8}{3} = -\frac{10}{3} + \frac{18}{3} \\ \frac{8}{3} = \frac{8}{3} & \frac{8}{3} = \frac{8}{3} \\ \text{(yes!)} & \text{(yes!)} \end{array}$$

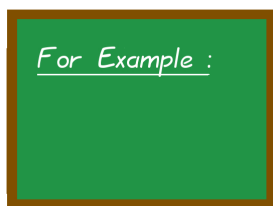
As you can see, plugging the coordinate point into each original equation produced the true, trivial statement $\frac{8}{3} = \frac{8}{3}$. This confirms that the point $(\frac{5}{3}, \frac{8}{3})$ is the intersection point of the two lines.



Assume you are given a system of linear equations and a coordinate pair (x, y) . To determine whether the coordinate is a solution to the system, you must plug the x - and y - values into *both* linear equations and check that a true mathematical expression follows from each. This will confirm that the two lines intersect at that coordinate. If only one, or neither, of the equations produces a true expression from the variable substitution, then the coordinate pair (x, y) is not a solution to the system.

Consider the linear system

$$\begin{cases} y = -x + 10 \\ y = 1.5x - 10 \end{cases}$$



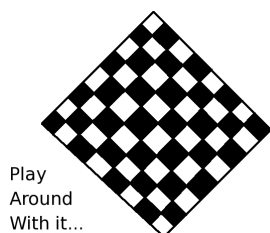
Is the coordinate pair $(8, 2)$ a solution to the system? To find out, you must plug the potential solution into both equations of the system.

$$\begin{array}{ll} y = -x + 10 & y = 1.5x - 10 \\ (2) = -(8) + 10 & (2) = 1.5(8) - 10 \\ 2 = -8 + 10 & 2 = 12 - 10 \\ 2 = 2 & 2 = 2 \\ \text{(yes!)} & \text{(yes!)} \end{array}$$

1-4-2

Both resulting expressions are true, which assures us that the point $(8, 2)$ is a solution. The fact that both true expressions are $2 = 2$ is a consequence of the linear equations being identically set up in the form $y = mx + b$.

For each bullet, determine whether the given coordinate pair solves the linear system.



1-4-3

- Does $(-1, -5)$ solve the system $\begin{cases} y = 3x - 2 \\ y = -x - 6 \end{cases}$?

[Answer: Yes, this is the solution to the system.]

- Does $(4, 1)$ solve the system $\begin{cases} y = -2x + 9 \\ y = 3x - 16 \end{cases}$?

[Answer: No, this is not the solution to the system.]

- Note, we really did have to check both equations in the second bullet, because $y = -2x + 9$ was satisfied while $y = 3x - 16$ was not satisfied. Since both equations must be satisfied, we reject $(4, 1)$ as a solution to the system.

Four boxes ago I presented the system of linear equations

$$\begin{cases} y = x + 1 \\ y = -2x + 6 \end{cases}$$

For Example :

1-4-4

It is in our interest to allow the equations in the system to *interact* with one another. Since we are looking for only one (x, y) -pair that solves the system, then the y -values must be equal.

Using this knowledge, we can rewrite the system as $x + 1 = y = -2x + 6$. Now, since $x + 1$ equals y and $-2x + 6$ also equals y , then it must be the case that $x + 1$ equals $-2x + 6$. Thus, we can write the equation $x + 1 = -2x + 6$. We solve this one-variable linear equation to obtain the x -value of our solution to the original system.

We will proceed with solving this linear system in the next box.

We now have a linear equation in one variable, $x + 1 = -2x + 6$. We can solve that easily, as follows:

$$\begin{aligned} x + 1 &= -2x + 6 \\ x + 2x &= 6 - 1 \\ 3x &= 5 \\ x &= \frac{5}{3} \end{aligned}$$

For Example :

We obtain the non-trivial expression $x = \frac{5}{3}$, which tells us that the two lines from the system of equations intersect at $x = \frac{5}{3}$. Next, to find the y -value of the point of intersection, we plug $x = \frac{5}{3}$ into one of the equations from the original system.

1-4-5

$$\begin{aligned} y &= \left(\frac{5}{3}\right) + 1 \\ y &= \frac{8}{3} \end{aligned}$$

The substitution of $x = \frac{5}{3}$ yielded $y = \frac{8}{3}$. In other words, the two lines $y = x + 1$ and $y = -2x + 6$ intersect at the point $\left(\frac{5}{3}, \frac{8}{3}\right)$.

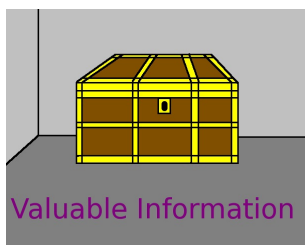
Checking the solution to a linear system is a matter of determining whether the lines share the same y -value at a particular x -value. In the previous box, we plugged our known x -value into one of the equations to find the corresponding y -value. To check your work, it is best practice to then plug the x -value into the other equation in the system, to see whether the same y -value is produced. If the resulting y -value is a match, then congratulations!—you have found the complete solution to your system.

Let's check that $\left(\frac{5}{3}, \frac{8}{3}\right)$ is a solution to the system by plugging $x = \frac{5}{3}$ into $y = -2x + 6$:

$$\begin{aligned} y &= -2\left(\frac{5}{3}\right) + 6 \\ y &= -\frac{10}{3} + \frac{18}{3} \\ y &= \frac{8}{3} \end{aligned}$$

Since we also obtained $y = \frac{8}{3}$ by plugging into $y = x + 1$, we know that the two lines intersect at $x = \frac{5}{3}$. Our solution is now confirmed.

**Check
Your
Work !!**

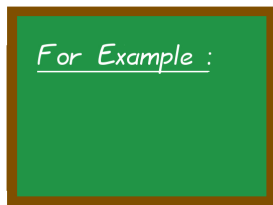


In general, when solving a system of linear equations where one variable is known, you need only plug that value into one of the two original equations to find the other variable. Then you can check your work by plugging the solution point into the unused equation to see whether a true expression results.

Let's try solving another system of linear equations. This time our system is

$$\begin{cases} y = 4x - 9 \\ y = 6 - x \end{cases}$$

If a solution exists, the y 's in both equations should be equal. Thus, we can collapse the system into a linear equation in x only and solve for x :



1-4-6

$$4x - 9 = 6 - x$$

$$4x + x = 6 + 9$$

$$5x = 15$$

$$x = 3$$

Having obtained $x = 3$, we then plug this value back into one of the two original equations to determine the corresponding y -value. Let's arbitrarily choose the first equation:

$$y = 4(3) - 9$$

$$y = 12 - 9$$

$$y = 3$$

Now we have a potential solution point for our system, $(3, 3)$.



To confirm that the coordinate $(3, 3)$ describes the intersection of the two lines in the above example, we plug the solution into the other, untested linear equation to check that a true mathematical expression is produced:

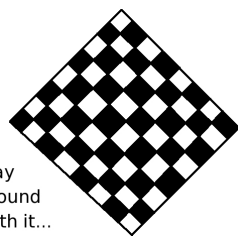
$$y = 6 - x$$

$$(3) = 6 - (3)$$

$$3 = 3$$

$$(yes!)$$

The validity of the resulting expression $3 = 3$ confirms that the second line contains the point $(3, 3)$, which we found to also be contained within the first line. Therefore, the two lines *do* intersect at $(3, 3)$.



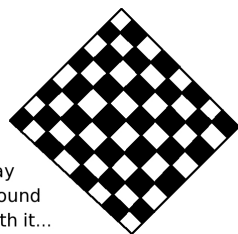
Play
Around
With it...

1-4-7

Find the point where the following two lines intersect:

$$\begin{cases} y = 0.2x - 24 \\ y = -0.5x + 11 \end{cases}$$

[Answer: The two lines intersect at the point $(50, -14)$.]



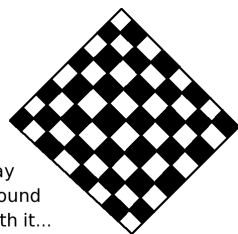
Play
Around
With it...

1-4-8

Find the point where the following two lines intersect:

$$\begin{cases} y = 41x + 58 \\ y = -17x + 29 \end{cases}$$

[Answer: The two lines intersect at the point $(-0.5, 37.5)$.]



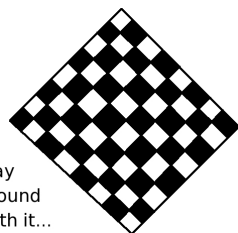
Play
Around
With it...

1-4-9

Find the solution to the linear system:

$$\begin{cases} y = -33x - 4 \\ y = 21x - 10 \end{cases}$$

[Answer: The approximate solution to the linear system is $x = 0.1\overline{11}$, $y = -7.6\overline{66}$.]



Play
Around
With it...

1-4-10

Find the solution to the system of linear equations:

$$\begin{cases} y = 5x - 30 \\ y = -3x + 23 \end{cases}$$

[Answer: The system is solved when $x = 6.625$ and $y = 3.125$.]



When we are given a system of linear equations, we first assume that a solution exists for the system. That is, we expect the lines expressed by the linear equations to intersect somewhere in the coordinate plane. Working from that assumption, we first try to find the value of one of the variables (so far we have found x first, but that need not be the case, as you will discover shortly). If a solution exists for the first variable, this confirms our assumption that the lines intersect. What we are about to show is that if the variable disappears while solving, we must concede that the lines either do not intersect or are not unique; in either of these peculiar scenarios, there is no single solution to the system.

So far, you have only seen systems of linear equations for which a single solution exists. However, I have mentioned the caveat that a solution may not exist for some systems. In fact, there is a third scenario—that there are an infinite number of solutions to a system of linear equations. We will now explore all three scenarios in detail.

Scenario One (Common): Single Solution

A system of two linear equations in standard form is

$$\begin{cases} y_1 = m_1x_1 + b_1 \\ y_2 = m_2x_2 + b_2 \end{cases}$$

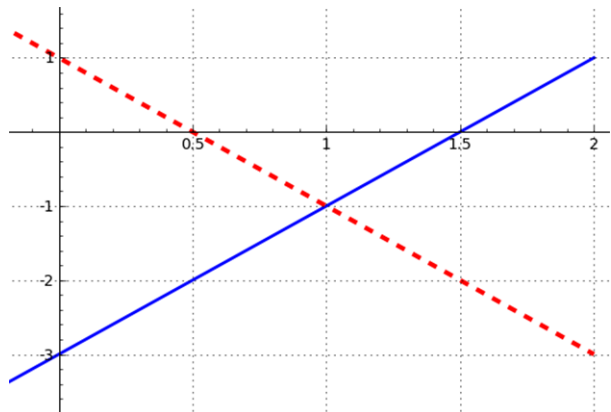
where m is the slope and b is the y -intercept of a line. If, for a system in standard form, $m_1 \neq m_2$ (the slopes are different), the system will have one unique solution. The graphical representation of the system will be two lines crossing one another at exactly one point on the plane.

The graph on the left plots the system of equations

$$\begin{cases} y = 2x - 3 \\ y = -2x + 1 \end{cases}$$

where it is clear that the lines have different slopes. Thus we would expect an intersection at only one point in the plane.

When a system of linear equations has exactly one solution, it is called an *independent system*.



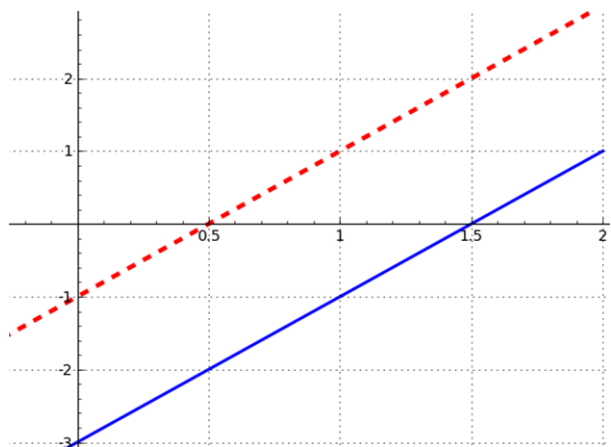
Scenario Two (Rare): No Solutions

See the model of a linear system in standard form, shown above. When $m_1 = m_2$ but $b_1 \neq b_2$, the system has no points of intersection (no solutions). This is because the lines are parallel but disjoint, running along forever in both directions without ever crossing.

Think of it this way: the lines have the same slope, but they cross the y -axis at different heights. Thus as you shift horizontally over the coordinate plane, both lines increase or decrease at the same rate; they are always separated vertically by the difference between b_1 and b_2 .

The graph on the left plots the system of equations

$$\begin{cases} y = 2x - 3 \\ y = 2x - 1 \end{cases}$$



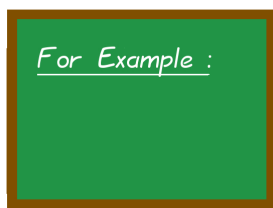


Looking at the previous box, we can see that the slopes of the lines match, but the y -intercepts do not. To be hyper-specific, the y -coordinate of any point on the blue solid line, for any particular value of x , will always be two less than the y -coordinate of the corresponding point on the red dashed line, for that same value of x . Therefore, this system has no solution.

When a system of linear equations has no solution, it is called an *inconsistent system*. If a system of linear equations is presented to you in standard form, it should be very apparent whether the lines are parallel or intersecting; simply compare the lines' slopes. However, there is another tell-tale sign that a system has no solutions: when solving for the first unknown variable, a nonsensical expression results rather than a true algebraic assignment. We will see an example of this in the next box.

Consider the system of linear equations

$$\begin{cases} y = 7x - 5 \\ y = 7x + 2 \end{cases}$$

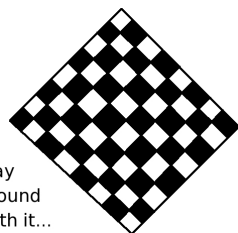


Since the slopes of the lines are equal but their y -intercepts are not, we can automatically deduce that the lines are parallel and thus do not intersect. However, let's collapse this system into a one-variable linear equation and see what we get when we try to solve it.

$$\begin{aligned} 7x - 5 &\stackrel{?}{=} 7x + 2 \\ 7x - 7x &\stackrel{?}{=} 5 + 2 \\ 0 &= 7 \end{aligned}$$

1-4-11

Our result $0 = 7$ is certainly false. And where did the variable x go? These errors arose for one reason only: we were trying to solve for an inconsistent system. We can conclude that the system has no solution.



Play
Around
With it...

1-4-12

Determine the number of solutions for each linear system. Then classify each system.

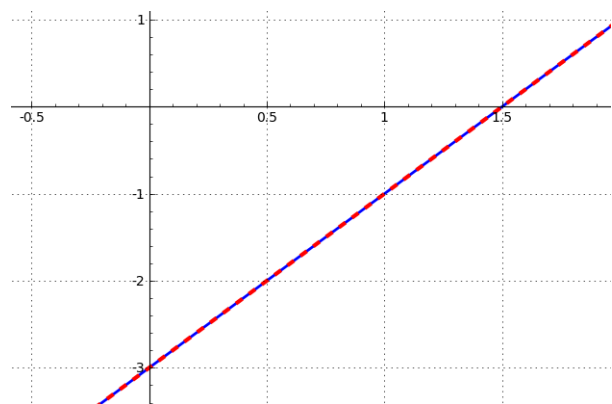
- $\begin{cases} y = 4x - 3 \\ y = 2 + 4x \end{cases}$ [Answer: The system has no solutions. It is an inconsistent system.]
- $\begin{cases} y = -5 - x \\ y = x - 10 \end{cases}$ [Answer: The system has one solution. It is an independent system.]



If I asked you to compare the two linear equations $y = 5x + 4$ and $y = 5x + 4$, you'd probably give me a sideways glance and tell me that "they are the same line, obviously." But what about $y = 5x + 4$ and $4y = 20x + 16$? Does it surprise you that these also describe the same exact line—that if you graphed these two linear equations on the coordinate plane, they would overlap perfectly?

Linear equations are, first and foremost, equations. And equations can be manipulated, including being scaled (multiplied) by a non-zero constant term. Notice that $4y = 20x + 16$ is just $y = 5x + 4$ with each of its terms multiplied by 4. Although the coefficients are larger, the linear relationship between y and x is preserved, which is what matters.

In short, if you can scale a linear equation—multiplying through by a constant term—so that it matches another linear equation, then the two equations are actually equivalent as objects (lines) on the coordinate plane.



Scenario Three (Rare): Infinitely Many Solutions

If, in a system of linear equations, one equation is a constant multiple of the other, then the two equations actually describe the same line. When graphed on the coordinate plane, the two lines will overlap one another.

That the lines overlap everywhere on the plane is another way of saying they intersect at all points. Therefore, a system of linear equations in which one equation is a constant multiple of the other has an infinite number of solutions: every point on either line is a solution to the system.

The graph to the left plots the system of equations

$$\begin{cases} y = 2x - 3 \\ 3y = 6x - 9 \end{cases}$$

We can see that the second equation is a multiple of the first (by a factor of 3), and that their corresponding lines overlap completely. Every point on the line $y = 2x - 3$ is a solution to the system.

When a system of linear equations has an infinite number of solutions, it is called a *dependent system*.

Find the point of intersection between the two lines described by the following equation:

$$\begin{cases} 2.5y = 5x - 12.5 \\ 1.5y = 3x - 7.5 \end{cases}$$

For Example :

First, I will get both equations into standard form, so that I can compare them. The top equation can be divided by 2.5, and the bottom equation can be divided by 1.5. I should reiterate that scaling an equation—either larger or smaller—does not alter the relationship between y and x in that equation.

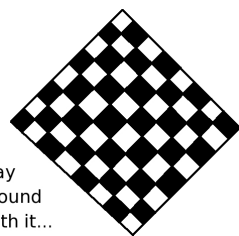
After scaling, the system of equations is rewritten as

$$\begin{cases} y = 2x - 5 \\ y = 2x - 5 \end{cases}$$

However, we can now see that these are the same line. Therefore, any point on the line $y = 2x - 5$ is a solution to the system. The system has an infinite number of solutions, represented by the linear equation $y = 2x - 5$.

1-4-13

You might now be tired of solving systems of two linear equations. However, we will see many word problems that are solved by these types of equations in Sections... , so it is beneficial to practice this skill just a little more.



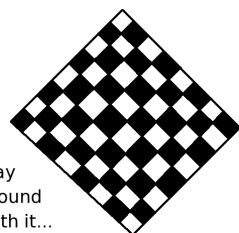
Play
Around
With it...

1-4-14

Solve the following system of linear equations, and then describe the system using one of the three scenarios introduced above:

$$\begin{cases} y = 3.5 - 3.3x \\ y = -3.3x + 3.3 \end{cases}$$

[Answer: The system has no solutions, and therefore is an inconsistent system.]



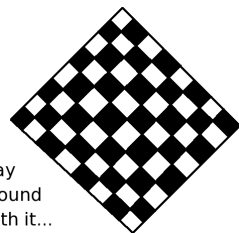
Play
Around
With it...

1-4-15

Solve the following system of linear equations, and then describe the system using one of the three scenarios introduced above:

$$\begin{cases} y = 1.3x + 30 \\ y = -\frac{1}{2}x + 18 \end{cases}$$

[Answer: The solution to this system is $x = -6.\overline{66}$, $y = 21.\overline{33}$. This system is independent.]



Play
Around
With it...

1-4-16

Solve the following system of linear equations, and then describe the system using one of the three scenarios introduced above:

$$\begin{cases} y = 100x + 25 \\ y = 25 + 100x \end{cases}$$

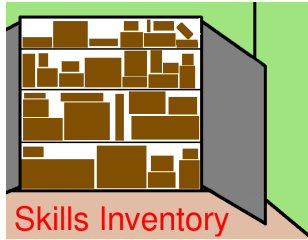
[Answer: This system has an infinite number of solutions, and is therefore dependent.]



So far you have only seen linear systems in which both equations were already in standard form. When this is the case, solving is only a matter of setting the polynomials in x equal to one another, solving for x , then using x to solve for y .

What if a system of equations is not presented in standard form? What if you were asked, for example, to find the intersection of $10 = 3x - 9y$ and $-2x = 5 + \frac{4}{5}y$? Luckily, there are two very popular algebraic methods available to solve systems where the equations are not in standard form; they are called the *Substitution Method* and the *Elimination Method*. The next module will focus on teaching you these methods, so that you can most efficiently solve any system of linear equations, no matter how complexly they are setup.

In this module we reviewed:



- how a system of linear equations in two variables represents a set of lines in the coordinate plane.
- the notion of a “solution” to a system of linear equations, and how it represents the point at which the lines cross in the plane.
- a simple method for solving systems of linear equations—or finding where two lines intersect—when the equations are in standard form.
- the characteristics of independent, inconsistent, and dependent linear systems.
- the idiosyncrasies that arise when trying to find solutions for an inconsistent or dependent system.
- the following vocabulary terms: system of linear equations, independent system, inconsistent system, dependent system.