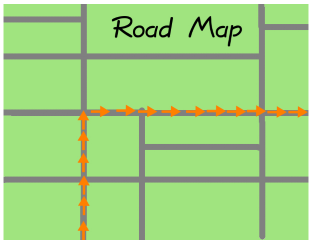
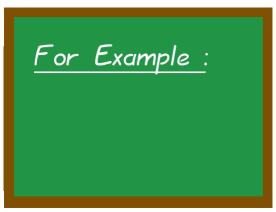
Module 1.5: Intersecting Two Lines, Part Two



In this module you will learn about two very common algebraic methods for intersecting two lines: the *Substitution Method* and the *Elimination Method*. Both methods are most useful when the equations in your system are not expressed in slope-intercept (or $y = mx+b$) form. Each method I present has its advantages; additionally, a given system of equations may lend itself more to one method for solving than the other. For these reasons it is important to learn both methods, so that you can use the tool that is best for you when solving any particular system of linear equations.



\# 1-5-1

I would like to solve the following system of equations:

$$\begin{cases} 2x - 3y = -4 \\ 4x + y = 20 \end{cases}$$

I notice that neither equation is in slope-intercept form. What I would like to do is manipulate one of the two equations to isolate a variable within it. The second equation, $4x + y = 20$, has a coefficient of 1 in front of the y, so it would be very easy to isolate the y here. Subtracting $4x$ from both sides of the second equation, I get $y = -4x + 20$.

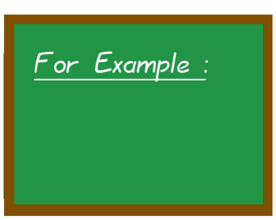
The next step is to generate a linear equation in only one variable, which is very easy to solve. I now have the equation $y = -4x + 20$. Now, I can substitute $(-4x + 20)$ for y in the first equation.

$$2x - 3(-4x + 20) = -4$$

That equation looks manageable, as it only contains the variable x after my substitution. We know how to solve linear equations in one variable, so let's do that.

$$\begin{aligned} 2x + 12x - 60 &= -4 \\ 14x &= 56 \\ x &= 4 \end{aligned}$$

We have managed to discover the value of x with just a few algebraic manipulations. In the next box, we will find the corresponding value of y and check that this process actually worked in producing the solution to the system.



\# 1-5-2

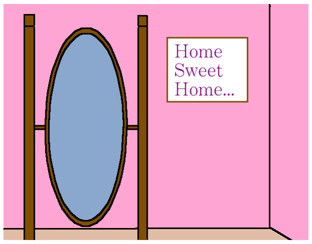
From the above process, where we used *substitution* to generate a solvable one-variable linear equation, we obtained the assignment $x = 4$. This looks like half the solution to the linear system. We will plug $x = 4$ into both original linear equations to see whether the results agree.

$$\begin{aligned} 2(4) - 3y &= -4 & \quad 4(4) + y &= 20 \\ 8 - 3y &= -4 & \quad 16 + y &= 20 \\ -3y &= -12 & \quad y &= 4 \\ y &= 4 & & \end{aligned}$$

The results agree. For each linear equation, $x = 4$ has the corresponding y-value $y = 4$. Therefore, both lines expressed by the system contain the point $(4, 4)$—so they must intersect there. The problem is solved.

Note, it was not necessary to plug $x = 4$ into both equations to find the value of y. Plugging into either one of the equations would have been sufficient—the fact that I obtained a value for x in the first place was evidence enough that a solution existed. As you can see, the value for y was the same in both equations.

If you find the solution to a linear system by working with only one of the two linear equations, you can check your work by plugging the solution into the other equation. If the equation is satisfied, then you can be sure that your solution is correct.

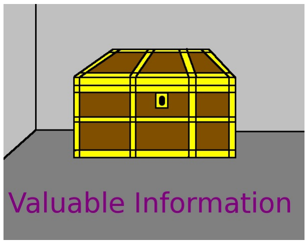


A Pause for Reflection...

This "substitution method" found the solution to a system of equations that was not expressed in slope-intercept form. I would like to formalize the method into a sequence of steps now, and caution you against the pitfalls that can arise when the method is misused.

This formalization of the method will be accompanied by an example, to illustrate the process. While describing the steps in the Substitution Method, I will be solving the following system of equations:

$$\begin{cases} 4x + 3y = 9 \\ 2y - x = 6 \end{cases}$$



The Substitution Method (Step 1)

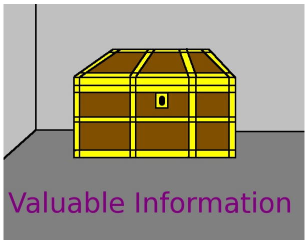
Among the two equations in the system, pick one from which you would like to isolate a variable. This is largely a judgement call on your part, as the amount of algebraic maneuvers you perform in this step is determined by the complexity of the equation you choose to manipulate, as well as the variable you choose to isolate in that equation.

- Working with the system described in the previous box, I would like to isolate the x from the second equation, because it already has 1 as its coefficient.

$$\begin{array}{rcl} & & 2y - x = 6 \\ \text{(becomes)} & \longrightarrow & 2y = 6 + x \\ \text{(becomes)} & \longrightarrow & 2y - 6 = x \end{array}$$

- I now have the equation $x = 2y - 6$, where x is described in terms of y. The system of equations now looks like this:

$$\begin{cases} 4x + 3y = 9 \\ 2y - 6 = x \end{cases}$$



The Substitution Method (Step 2)

Having isolated a variable from one of the two linear equations, substitute the relation for that variable into the *other*(!) equation to obtain a linear equation in one variable.

- I will substitute $(2y - 6)$ for x in the first linear equation, $4x + 3y = 9$.

$$4(2y-6) + 3y = 9$$
$$8y - 24 + 3y = 9$$
$$11y - 24 = 9$$

- By substituting for x, I have turned the two-variable linear equation $4x + 3y = 9$ into the single-variable linear equation $11y - 24 = 9$.

I must emphasize strongly, that when performing Step 2 of the Substitution Method, you plug into the equation that you *did not* use in Step One.

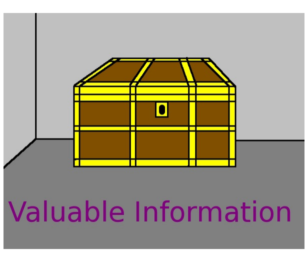
If you isolate a variable in an equation and plug back into the original equation, this will always result in a trivial expression, from which you can derive no solution value for either variable. You must plug into the other (previously unused) equation to make progress in solving the system.

To illustrate this point, try substituting $(2y-6)$ for x back into the equation $2y - x = 6$.

$$2y - (2y - 6) = 6$$
$$2y - 2y + 6 = 6$$
$$6 = 6$$

Both variables have been cancelled out, and we are left with the trivial expression $6 = 6$, which does not aid us in solving the original system of equations.

This canceling-out of variables will always occur when you plug back into the same equation. Therefore, you must always substitute into the "other" equation to obtain a useful solution value.



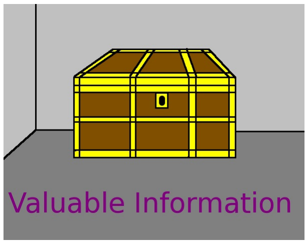
The Substitution Method (Step 3)

Once you obtain a linear equation in one variable, you then solve that equation. This yields a partial solution to the linear system.

- I will now solve $11y - 24 = 9$ for the variable y.

$$11y - 24 = 9$$
$$11y = 33$$
$$y = 3$$

- I have found the partial solution $y = 3$.



The Substitution Method (Step 4)

You now have one of the values of the (x, y) pair that describes the intersection point of the system. Plug this value into either of the two original linear equations within the system, to obtain a linear equation in terms of the still unknown variable. Solve this equation to find the solution value of the unknown variable.

Whichever equation you select is another judgment call that you must make. Either way the result will be the same, if you have followed the steps properly so far. As before, try to make your work easier by choosing whatever equation will be simpler to work with.

- I have the partial solution $y = 3$. I will plug this value into the linear equation $2y - x = 6$ to find the solution value of x.

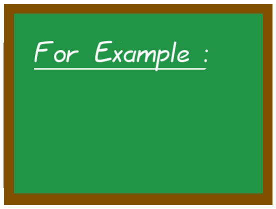
$$\begin{aligned} 2(3) - x &= 6 \\ 6 - x &= 6 \\ 6 - 6 &= x \\ 0 &= x \end{aligned}$$

- I get $x = 0$ as the result. Combining this with the value I found for y earlier, I determine that the two lines intersect at the point $(0, 3)$.

Are my results from the Substitution Method correct? Is the solution to the system $(0, 3)$? I should plug these coordinates into the other equation, $4x + 3y = 9$, to check.

$$\begin{gathered} 4(0) + 3(3) \stackrel{?}{=} 9 \\ 9 = 9 \\ \text{(yes!)} \end{gathered}$$

A trivial yet true expression follows, therefore I conclude that I have found the solution for the linear system.



\# 1-5-3

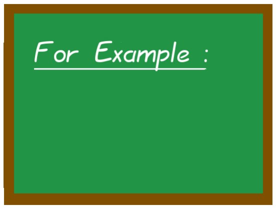
Let's solve the following linear system using the Substitution Method:

$$\begin{cases} 6x - 2y = -6 \\ -4x + 3y = 4 \end{cases}$$

Neither equation has a variable with 1 as its coefficient. However, each coefficient in the first equation is a multiple of 2, which will make it easy to isolate y from this equation.

$$\begin{aligned} 6x - 2y &= -6 \\ 6x &= 2y - 6 \\ 6x + 6 &= 2y \\ 3x + 3 &= y \end{aligned}$$

This completes Step One of the Substitution Method.



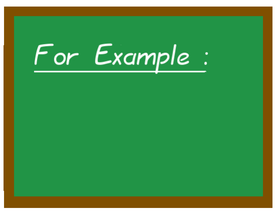
\# 1-5-4

Continuing from the previous box, we will substitute the expression $(3x+3)$ for y into the second equation, to obtain

$$\begin{aligned} -4x+3(3x+3) &= 4 \\ -4x+9x+9 &= 4 \\ 5x+9 &= 4 \\ 5x &= -5 \\ x &= -1 \end{aligned}$$

$x=-1$ is a partial solution to the linear system.

Here I have completed Steps Two and Three of the Substitution Method.



\# 1-5-5

I found a single value for the variable $x=-1$. This tells me that the system is *independent*, meaning the two lines intersect at only one point. Finally, I plug $x=-1$ into either equation to determine the solution value of y. I arbitrarily choose to work with the first equation.

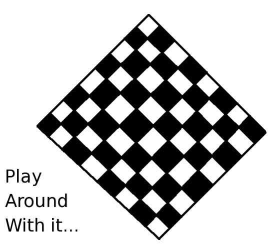
$$\begin{aligned} 6(-1)-2y &= -6 \\ -6-2y &= -6 \\ -2y &= 0 \\ y &= 0 \end{aligned}$$

The solution I obtain for the linear system is $x=-1,\ y=0$. In the next box I will check my work.

Do the coordinates $(-1,0)$ solve the system from the previous three boxes? I will plug these values into the equation $-4x+3y=4$ to check my work.

$$-4(-1)+3(0) \stackrel{?}{=} 4$$
$$4=4$$
$$\text{(yes!)}$$

I conclude that I have found the intersection point of the two lines described by the system.



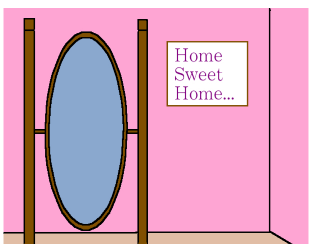
\# 1-5-6

Solve the following systems of equations using the Substitution Method. Be sure to check your work.

- Solve $\begin{cases} 10x-4=3y \\ -2y+8=4x \end{cases}$ [Answer: The solution is $x=1,\ y=2$.]
- Solve $\begin{cases} 5+9x=-2y \\ 8x+3y=9 \end{cases}$ [Answer: The solution is $x=-3,\ y=11$.]
- Solve $\begin{cases} 4+3y=4x \\ -2y+6=-7x \end{cases}$ [Answer: The solution is $x=-2,\ y=-4$.]

In the above Playbox, for each of the bullets we could have expressed our answer as "The lines intersect at (a, b)" instead of "The solution is $x = a,\ y = b$."

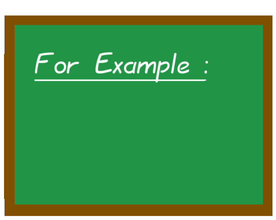
For example, for the middle bullet we could say "The lines intersect at $(-3, 11)$."



A Pause for Reflection. . .

In the past few pages, we have used the Substitution Method to solve independent systems of equations: for each system, there was only one solution, indicating that the lines intersected at only one point on the plane.

You might expect that difficulties arise when trying to solve inconsistent systems (parallel lines) or dependent systems (the same line) using the Substitution Method. This intuition would be correct—the same errors occur for these particular scenarios as was discussed in the previous module (at considerable length). We will now explore examples of each scenario.



1-5-7

Try to solve the inconsistent system of linear equations

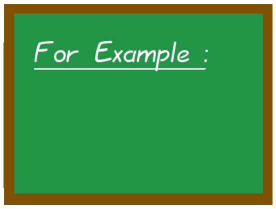
$$\begin{cases} 6x - 4y = 5 \\ 7 + 2y = 3x \end{cases}$$

Isolating x in the second equation, we get the statement $x = \frac{7}{3} + \frac{2}{3}y$. Substituting the right-hand side for x into the first equation produces

$$\begin{aligned} 6(\tfrac{7}{3} + \tfrac{2}{3}y) - 4y &= 5 \\ 14 + 4y - 4y &= 5 \\ 14 &= 5 \end{aligned}$$

Alas, the variable we were solving for has disappeared, and we are left with a nonsensical expression. Since it is not the case that $14 = 5$, we know that this system is inconsistent.

Rule: If during Step 2 of the Substitution Method, a nonsensical (false) statement is produced instead of a linear equation in one variable, then you can be sure that the system is inconsistent and therefore has no solutions.



1-5-8

Try to solve the dependent system of linear equations

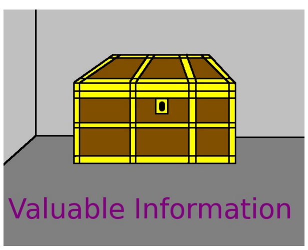
$$\begin{cases} 8 = 5x - 2y \\ 7.5x = 3y + 12 \end{cases}$$

Isolating y in the first equation, we get the statement $y = 2.5x - 4$. Substituting the right-hand side for y into the second equation produces

$$\begin{aligned} 7.5x &= 3(2.5x - 4) + 12 \\ 7.5x &= 7.5x - 12 + 12 \\ 0 &= 0 \end{aligned}$$

This time, the variable we were solving for has disappeared, and the expression left behind is trivial but undeniably true. (We know that $0 = 0$.) As you read earlier, this will always occur when trying to solve a dependent system.

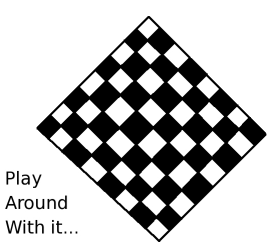
RULE: If during Step 2 of the Substitution Method, a trivial (but true) algebraic expression is produced instead of a linear equation in one variable, then you can be sure that the system is dependent and has infinitely many solutions. In this case, the solution can be expressed as either linear equation reduced to a convenient form.



In order for us to deduce that a system of equations has a unique solution, the process of solving for the first unknown must must produce an answer, such as $x = 4$ or $y = -6$.

If, instead, a nonsensical expression like $5 = -3$ occurs, this indicates that the system describes a pair of parallel lines, for which no intersection point exists.

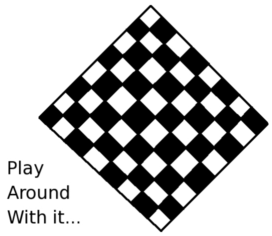
Finally, if solving for the first unknown variable yields a true but trivial expression, like $1 = 1$ or even $0 = 0$, then we can be sure that the system of equations contains duplicates of the same line. In this scenario, the solution is the entire line described by either linear equation.



1-5-9

Solve the following systems of equations using the Substitution Method:

- Solve $\begin{cases} -4 + 3y = -11x \\ 5y + 9x = 2 \end{cases}$ [Answer: The solution is $x = 0.5,\ y = -0.5$.]
- Solve $\begin{cases} 5y = 12x + 16 \\ 1.25y - 3x = 6 \end{cases}$ [Answer: The system is inconsistent. It has no solutions.]
- Solve $\begin{cases} -2y + 4 = 3x \\ y + 5x = -5 \end{cases}$ [Answer: The solution is $x = -2,\ y = 5$.]



1-5-10

Solve the following systems of equations using the Substitution Method:

- Solve $\begin{cases} 3y + 2 = 2x + 3 \\ 3 - 4x = 5 - 6y \end{cases}$ [Answer: The system is dependent. The solution is the set of all (x, y) pairs that satisfy the equation $y = \frac{2}{3}x + \frac{1}{3}$.]

- Solve $\begin{cases} -2 + 3y = -6x \\ 6y + 4x = 3 \end{cases}$ [Answer: The lines intersect at $\left(\frac{1}{8}, \frac{5}{12}\right)$.]

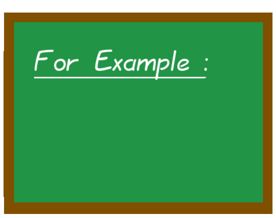
- Solve $\begin{cases} 5x - 5 = 9y \\ 3y + 10 = x \end{cases}$ [Answer: The solution is $x = -12.5,\ y = -7.5$.]

Remember when, early in this module, we were given a system of equations in slope-intercept form and we solved for x by equating the polynomials in x? The setup looked like this:

$$m_1 x + b_1 = m_2 x + b_2$$

This process was actually a perfect implementation of the Substitution Method. We took one relation to y and substituted it for the other y, to obtain a linear equation in x. Since we didn't have to manipulate either equation, however, we effectively skipped the first step of the Substitution Method. It was good practice for us to solve these simpler systems first, before moving on to more complex systems of equations.

At this time I will introduce to you the *Elimination Method*, the second of two algebraic methods you can use to solve a system of linear equations.



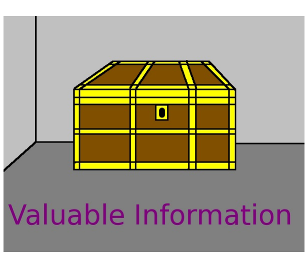
1-5-11

Suppose I am presented with the system of linear equations

$$\begin{cases} 4x + 19y = 2 \\ 3x + 19y = 4 \end{cases}$$

and I would like an alternative to the Substitution Method to solve this system. Perhaps my reasoning is that I do not want to work with awkward fractions (in particular, 19ths), which would happen if I tried to isolate any of the variables, due to the coefficient 19.

Anyway, there is another method of solving available to me, called "Elimination," but first I must review a basic property of algebra.



Casually, you can always add two equations together, or subtract one equation from another. More formally, this means:

1. if $a = b$ and $c = d$, then $a + c = b + d$.
2. if $a = b$ and $c = d$, then $a - c = b - d$.
3. if $a = b$ and $c = d$, then $c - a = d - b$.

For Example :

\# 1-5-12

Continuing with the example from above, I look at my linear system again,

$$\begin{cases} 4x + 19y = 2 \\ 3x + 19y = 4 \end{cases}$$

and notice that both equations share the same $ax + by = c$ structure. From the Valuable Information box above, I know that I can subtract one equation from the other to generate a new, true equation. But what advantage would that serve me?

In both equations the coefficient of y is 19. If I subtract the second equation from the first, the resulting equation would have a y term with a 0 coefficient—meaning the y would be *eliminated*, and I would be left with a linear equation in x.

$$\begin{array}{rrcr} & 4x + 19y = & & 2 \\ - & 3x + 19y = & & 4 \\ \hline & x + \quad\; = & & -2 \end{array}$$

Using this process, where I *eliminated* one variable by combining the equations through addition/subtraction, I was able to solve for the other variable. So far, my partial solution is $x = -2$.

For Example :

\# 1-5-13

At this point I have $x = -2$ and the original system of equations. I can solve for y by plugging my value of x into either equation—by now we know that both equations will yield the same solution value for y, since a value for x was obtained.

The only difference between the Elimination Method and the Substitution Method, then, is the first step—discovering the value of the first unknown. After that is achieved, the process for solving is similar.

I will complete this example by solving for y. I will plug $x = -2$ into the equation $4x + 19y = 2$.

$$\begin{aligned} 4(-2) + 19y &= 2 \\ -8 + 19y &= 2 \\ 19y &= 10 \\ y &= \tfrac{10}{19} \end{aligned}$$

I arrive at the solution $\left(-2, \frac{10}{19}\right)$ for the system.

For Example :

\# 1-5-14

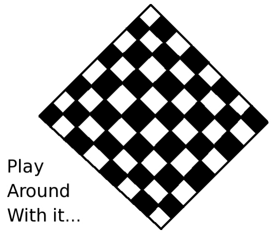
If, between the two equations in a system, one variable shares the same coefficient but has opposite signs, for example

$$\begin{cases} 2x + 10y = 4 \\ 5x - 10y = 1 \end{cases}$$

then I could just as easily add the two equations together, rather than subtract them. This process would still eliminate the variable, leaving me with a linear equation in terms of the other variable.

$$\begin{array}{rrcr} & 2x + 10y = & & 4 \\ + & 5x - 10y = & & 1 \\ \hline & 7x + \quad\; = & & 5 \end{array}$$

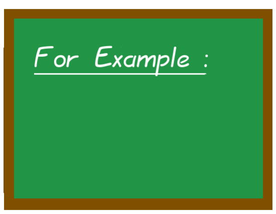
The challenge is recognizing *which* variable can be easily eliminated, as well as what operation between the equations will do this for you.



1-5-15

Mimicking the steps I performed above, solve the following systems of linear equations. (The first step for each system should be eliminating one variable by adding or subtracting the equations.)

- Solve $\begin{cases} 2+3x=4y \\ 8+3x=5y \end{cases}$ [Answer: The solution is $x=\frac{22}{3},\ y=6$.]
- Solve $\begin{cases} 2y-10x=-2 \\ -2y+5x=-10 \end{cases}$ [Answer: The solution is $x=\frac{12}{5},\ y=11$.]
- Solve $\begin{cases} x-6=-5y \\ 2x-2=-5y \end{cases}$ [Answer: The solution is $x=-4,\ y=2$.]



1-5-16

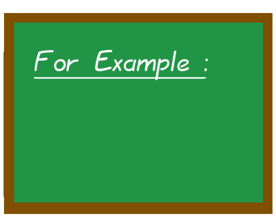
Let's try a harder problem using the method of elimination. I am trying to find the intersection point of the two lines represented by the system

$$\begin{cases} 8y-10=-3x \\ -2+6x=-4y \end{cases}$$

I notice two things about this system: First, that the terms in each equation are not ordered in an identical manner. Let me rewrite the system so that the variables and constant terms are vertically aligned:

$$\begin{cases} 3x-10=-8y \\ 6x-2\ \ =-4y \end{cases}$$

This system is not yet reducible through addition or subtraction, but I am getting closer.



1-5-17

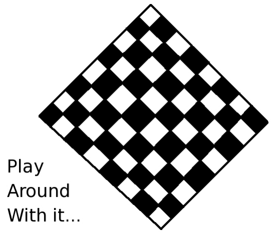
My second observation is this: even though the equations are in a consistent form, there is no variable that shares the same coefficient between the two equations. However, the coefficient for x in the top equation is exactly half that of the coefficient for x in the bottom equation. If I multiply the top equation by 2...

$$\begin{cases} 6x-20=-16y \\ 6x-2=-4y \end{cases}$$

Now, subtracting the bottom equation from the top equation produces $-18=-12y$, which gives me $y=\frac{3}{2}$. Plugging this value for y into the top equation in the original system,

$$\begin{aligned} 8(\tfrac{3}{2})-10&=-3x \\ 12-10&=-3x \\ -\tfrac{2}{3}&=x \end{aligned}$$

After plugging the pair $x=-\frac{2}{3},\ y=\frac{3}{2}$ into the other (bottom) equation in the original system, I determine that I have indeed found the intersection point between the two lines.



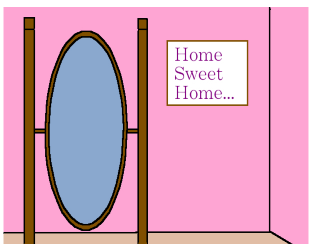
1-5-18

In the example above, after rewriting the linear system as

$$\begin{cases} 3x - 10 = -8y \\ 6x - 2 = -4y \end{cases}$$

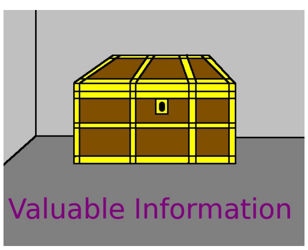
I then multiplied the top equation by 2 to make the coefficients for x agree. However, it is equally convenient to scale one of the equations so that the y terms can be eliminated. Describe how you would rewrite this system to eliminate the y's, then check that you arrive at the same solution after finding the two unknowns.

[Answer: One approach would be to multiply the bottom equation by 2, making the coefficient for y be -8 in both equations. Then you would subtract one equation from the other to eliminate the y's. (You could also multiply the bottom equation by -2 and add the two equations together.) This will yield the same solution as was found in the previous box.]



A Pause for Reflection...
At this point I will formalize the second algebraic process for solving systems of equations, known as the *Elimination Method*. While describing the method, I will illustrate its steps by solving the following system of equations:

$$\begin{cases} 2 + 3x = -4y \\ 7x - 4 = -10y \end{cases}$$

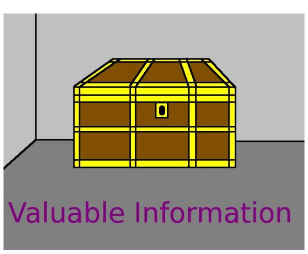


The Elimination Method (Step 1)

The first thing you want to do to a linear system is algebraically rearrange both equations so that the variable and constant terms are vertically aligned.

- For the example system described in the previous box, I want to rearrange each equation into the form $ax + by = c$.

$$\begin{cases} 3x + 4y = -2 \\ 7x + 10y = 4 \end{cases}$$

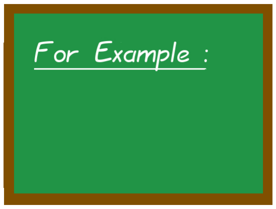


The Elimination Method (Step 2)

Once the equations are in a consistent arrangement, determine how you can scale one or both equations to equate the coefficient values of one of the variables. The sign in front of each coefficient is not important yet.

How you scale the equations is largely a judgement call. If the corresponding coefficients for a variable (x or y) are not already integral multiples of one another, you need to find the least common multiple between the two coefficients, m, and scale both equations so that both coefficients are equal to m.

In the next box we will demonstrate how to carry out this step.



1-5-19

The Elimination Method (Step 2)

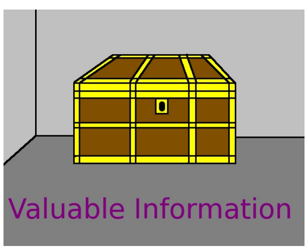
- Looking at my system

$$\begin{cases} 3x + 4y = -2 \\ 7x + 10y = 4 \end{cases}$$

I decide that I want to eliminate the x's from both equations. 3 does not evenly divide 7, so I must find their least common multiple—that would be 21.

- Now I want to scale each equation so that the coefficient in front of the x term is 21. This requires me to multiply the top equation by 7 and the bottom equation by 3, transforming the system into

$$\begin{cases} 21x + 28y = -14 \\ 21x + 30y = 12 \end{cases}$$



The Elimination Method (Step 3)

Once you have manipulated the system to get a variable's coefficients to agree *in number* (ignoring if they are positive or negative), you have two options:

i. If the corresponding coefficients have the *same* sign, then you subtract one equation from the other to eliminate the target variable.

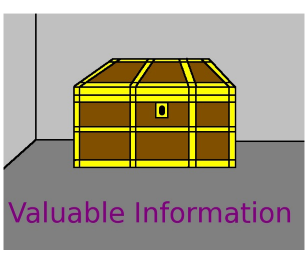
ii. If the corresponding coefficients have the *opposite* sign, then you add the two equations to eliminate the target variable.

Finally, reduce the resulting linear equation to solve for the variable that remains.

- Since both coefficients of y in my system have the same sign, I will subtract the top equation from the bottom.

$$\begin{array}{rrr} & 21x + 30y = & 12 \\ - & 21x + 28y = & -14 \\ \hline & 2y = & 26 \end{array}$$

- The expression $2y = 26$ is reduced to the partial solution $y = 13$.



The Elimination Method (Step 4)

You have found the value of one of the variables. Plug this value into either of the linear equations in the original system to determine the value of the other variable. You will then have found your solution point. Be sure to check your work by plugging the solution values into the other equation in the system.

- Plugging $y = 13$ into either of the original linear equations gives me $x = -18$. Therefore, the solution to the system is $(-18, 13)$. I encourage you to check this solution.

For Example :

1-5-20

When my system was set up as

$$\begin{cases} 3x + 4y = -2 \\ 7x + 10y = 4 \end{cases}$$

I decided to eliminate the x terms to find the value of y. But what if I had done the exact opposite? Let's say I instead want to eliminate the y's from both equations, to solve for x.

The least common multiple of the coefficients for y (4 and 10) is 20. So let me scale each equation so that the y coefficients become 20. This requires me to multiply the top equation by 5 and the bottom equation by 2, yielding the adjusted system

$$\begin{cases} 15x + 20y = -10 \\ 14x + 20y = 8 \end{cases}$$

I will now subtract the bottom equation from the top equation to eliminate the y terms.

$$\begin{array}{crr} & 15x + 20y = & -10 \\ - & 14x + 20y = & 8 \\ \hline & x + \quad\; = & -18 \end{array}$$

I still obtain -18 as the x-value. Plugging $x = -18$ into the original first equation, $3x + 4y = -2$, I get $y = 13$. I found the same solution point as before.

For Example :

1-5-21

Some students are not comfortable with figuring out the "least common multiple." Okay, what can you do then? One option is to simply use substitution, which is fine because substitution and elimination will always produce the same result. Another option is to multiply each equation by the coefficient found in the other equation. This is actually a bit wasteful, but it always works. For example, to equate the y's,

$$\begin{array}{rcll} 3x + 4y = -2 & \longrightarrow & 30x + 40y = -20 & \text{(we multiplied by 10)} \\ 7x + 10y = 4 & \longrightarrow & 28x + 40y = 16 & \text{(we multiplied by 4)} \end{array}$$

Subtracting the bottom scaled equation from the top scaled equation,

$$\begin{array}{crr} & 30x + 40y = & -20 \\ - & 28x + 40y = & 16 \\ \hline & 2x + \quad\; = & -36 \end{array}$$

This reduces to $x = -18$, from which you can proceed to find y by plugging $x = -18$ back into either equation in the system.

In the previous box, I had used the strong word "wasteful." Perhaps you're wondering what I mean by that. Consider two students examining the following problem:

$$\text{Solve} \begin{cases} 100x + 360y = 5860 \\ 45x + 180y = 2835 \end{cases}$$

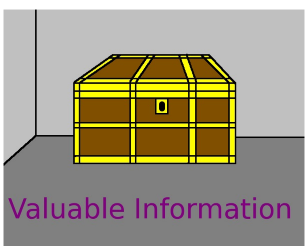
Looking at the y coefficients, one student might notice that $2 \times 180 = 360$, and endeavor to solve via a smooth and painless Elimination-Method experience. However, the other student might try the multiplication trick described in the previous box. If he wishes to eliminate x, then the top equation would be multiplied by 45 and the bottom by 100, to obtain

$$\begin{cases} 4500x + 16,200y = 263,700 \\ 4500x + 18,000y = 283,500 \end{cases}$$

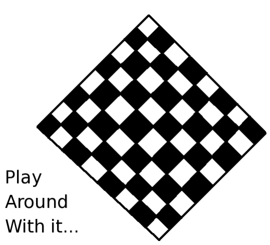
which is fairly nasty. If the second student were instead eliminating y, then the top equation would be multiplied by 180 and the bottom by 360, to obtain

$$\begin{cases} 18,000x + 64,800y = 1,054,800 \\ 16,200x + 64,800y = 1,020,600 \end{cases}$$

which I'm sure we can all agree is even nastier. Therefore, it is usually worthwhile to either find small numbers to multiply each equation by, or if that does not work for some reason, then to use the Substitution Method instead.



No matter which variable you choose to eliminate when performing Step 1 of the Elimination Method, you will achieve the same correct solution, as long as a solution exists (meaning, as long as the system is independent).

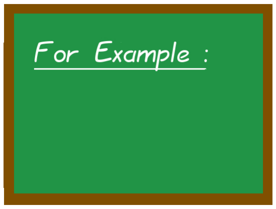


1-5-22

Solve the following systems of equations using the Elimination Method.

- Solve $\begin{cases} 5x - 8y = 8 \\ -6y + 3x = 15 \end{cases}$ [Answer: The lines intersect at $(-12, -8.5)$.]
- Solve $\begin{cases} 9 - 4x = 12y \\ 5 + 2y = -5x \end{cases}$ [Answer: The lines intersect at $(-1.5, 1.25)$.]

As is becoming customary by now, I must address what drawbacks arise during the Elimination Method when trying to solve an inconsistent or dependent system.



1-5-23

Find where the following two lines intersect, using the Elimination Method.

$$\begin{cases} 5y = 6x + 10 \\ 12x = 10y + 2 \end{cases}$$

- First I rearrange the equations to vertically align similar terms.

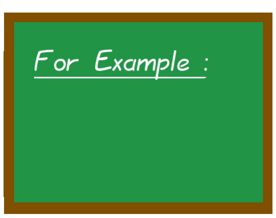
$$\begin{aligned} 5y &= \quad 6x + 10 \\ -10y &= -12x + 2 \end{aligned}$$

- Looking at the coefficients of the y terms, I realize that 10 is a multiple of 5. To eliminate the y terms, I must multiply the top equation by 2.

$$\begin{aligned} 10y &= \quad 12x + 20 \\ -10y &= -12x + 2 \end{aligned}$$

- Since the y coefficients have opposite signs, I add the two equations. This results in the nonsensical expression $0 = 22$, which only occurs when the system is inconsistent.

Rule: When trying to solve an inconsistent system (parallel lines, having no solution), eliminating one of the variables will inadvertently eliminate both variables, leaving you with a nonsensical algebraic equation.



1-5-24

Find where the following two lines intersect, using the Elimination Method.

$$\begin{cases} 4y = 2 + 3x \\ 3 + 4.5x = 6y \end{cases}$$

- First, I rearrange the equations to vertically align similar terms.

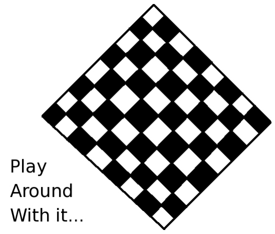
$$\begin{aligned} 4y &= \quad 3x + 2 \\ 6y &= 4.5x + 3 \end{aligned}$$

- Looking at the coefficients of the x terms, I realize that 9 is the least common multiple of 3 and 4.5. To eliminate the x terms, I must multiply the top equation by 3 and the bottom equation by 2.

$$\begin{aligned} 12y &= 9x + 6 \\ 12y &= 9x + 6 \end{aligned}$$

- By now it is obvious that the two equations are equivalent. Subtracting one from the other yields the trivial statement $0 = 0$, which only occurs with dependent systems.

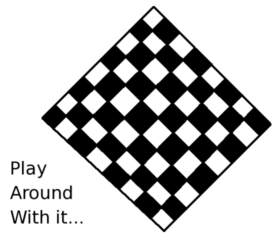
Rule: When trying to solve a dependent system (duplicates of the same line, having infinite solutions), eliminating one of the variables will inadvertently eliminate all terms, leaving you with the trivial algebraic equation $0 = 0$.



1-5-25

Solve the following systems of equations using the Elimination Method:

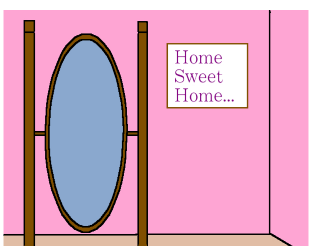
- Solve $\begin{cases} -4+3y=-6x \\ 6y+5x=3 \end{cases}$ [Answer: The lines intersect at $\left(\frac{5}{7}, -\frac{2}{21}\right)$.]
- Solve $\begin{cases} 5y+4=2x \\ 12x-2=8y \end{cases}$ [Answer: The solution is $x=-0.5,\ y=-1$.]
- Solve $\begin{cases} 2y-9=3x \\ 6x+18=4y \end{cases}$ [Answer: The system is dependent. The solution is the set of all (x,y) pairs that satisfy the equation $y=1.5x+4.5$.]



1-5-26

Solve the following systems of equations using the Elimination Method:

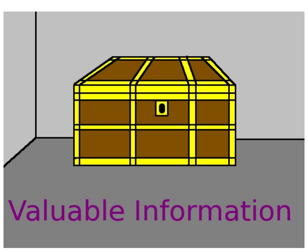
- Solve $\begin{cases} -4+3y=-10x \\ 6y+8x=2 \end{cases}$ [Answer: The point of intersection is $\left(\frac{1}{2}, -\frac{1}{3}\right)$.]
- Solve $\begin{cases} 3y=7x+10 \\ 4.5y-10.5x=6 \end{cases}$ [Answer: The system has no solutions. It is inconsistent.]
- Solve $\begin{cases} -3y+5=4x \\ 2y+5x=-6 \end{cases}$ [Answer: The solution is $x=-4,\ y=7$.]



A Pause for Reflection...

The question remains, which method—Substitution or Elimination—is preferred when intersecting two lines? The simple answer is, whatever you feel more comfortable with. Whereas the Substitution Method may appeal to you because it avoids the trouble of scaling the individual linear equations, the Elimination Method provides the advantage of quickly determining the value of the first unknown once the "elimination" step is complete.

No method is "easier" for all students than the other, so you must base your decision on your personal preference and the initial setup of the system. Either way, you are guaranteed to find the correct solution (if one exists) every time if you follow the methods properly. This is an even stronger guarantee if you check your work when you are done.

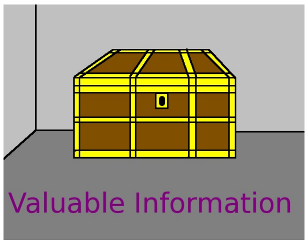


This has been a long module. It is helpful if we conclude with a recap of the techniques required in each of the two algebraic methods for intersecting two lines.

The Substitution Method

1. Isolate one of the variables in one of the linear equations.
2. Substitute that variable's relation into the *other* linear equation.
3. Solve the resulting one-variable linear equation.
4. Having found the solution value for one of the variables, plug that value into either original equation and solve for the other unknown variable.

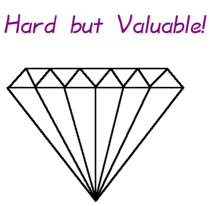
*5. Check your work by plugging the final solution into each equation of the original problem.



The Elimination Method

1. Rearrange both equations so that like terms are vertically aligned.
2. Decide how to scale the equations so that one variable has a common coefficient value, then multiply the equations accordingly.
3. Add or subtract the scaled equations to eliminate one of the variables completely, then solve the resulting linear equation in one variable.
4. Having found the solution value for one of the variables, plug that value into either original equation and solve for the other unknown variable.

*5. Check your work by plugging the final solution into each equation of the original problem.



There is actually one more algebraic method to solving linear systems, called Cramer's Rule. It is hard to memorize and very accident-prone. Therefore, I do not recommend that you use it, unless you are writing a computer program and need to intersect thousands, millions, or even billions of pairs of lines. This actually does happen in Video Game Design. For example, when walking through a virtual dungeon, the walls will form a maze, and the computer must calculate where the lines hit each other in order to be able to draw a lifelike maze. In all probability, you will never need to do this yourself.

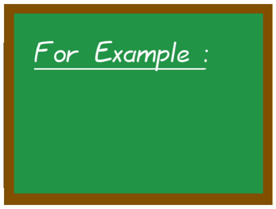
Cramer's Rule: if $ad - bc \neq 0$ then the system of linear equations

$$\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$$

has a unique solution, located at

$$\begin{aligned} x &= \frac{rd - bs}{ad - bc} \\ y &= \frac{as - rc}{ad - bc} \end{aligned}$$

On the other hand, if $ad - bc = 0$, then there are either infinitely many solutions or no solutions.



1-5-27

Of course, I should show you just one example of Cramer's Rule, so you can see how it actually works. Consider the system

$$\begin{cases} 11x + 19y = 23 \\ 29x + 31y = 17 \end{cases}$$

Plugging the coefficients to find x's value,

$$x = \frac{(23)(31) - (19)(17)}{(11)(31) - (19)(29)} = \frac{713 - 323}{341 - 551} = \frac{390}{-210} = \frac{39}{-21} = -\frac{13}{7}$$

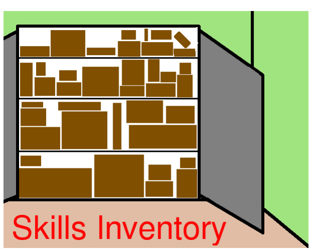
Now doing the same for y,

$$y = \frac{(11)(17) - (23)(29)}{(11)(31) - (19)(29)} = \frac{187 - 667}{341 - 551} = \frac{-480}{-210} = \frac{-48}{-21} = \frac{16}{7}$$

I invite you to check that $x = -\frac{13}{7}$, $y = \frac{16}{7}$ is indeed a solution.

In the 1950s and early 1960s, many algebra textbooks would include Cramer's Rule. Over the years, it became clear that although the formula is hard to memorize, it is not impossible. Nonetheless, there are so many opportunities for making an error by putting the wrong coefficient in one of the several spots in the formula, that this method almost never results in a student getting the question correct. Accordingly, the mathematics community wisely dropped Cramer's Rule from algebra textbooks. In the computer era, Cramer's Rule has been "born again" because computers do not make arithmetic mistakes.

In all fairness, it should be noted that there is another version of Cramer's Rule for linear systems of equations with 3-variables and 3-unknowns, 4-variables and 4-unknowns, or even larger systems. That version of Cramer's Rule is useful, practical, and interesting, but involves matrices, so I cannot describe that to you yet.



In this module we reviewed:

- the Substitution algebraic method for finding where two lines intersect.
- the Elimination algebraic method for finding where two lines intersect.
- the computational advantages to each algebraic method.
- the following vocabulary terms: Substitution Method, Elimination Method.