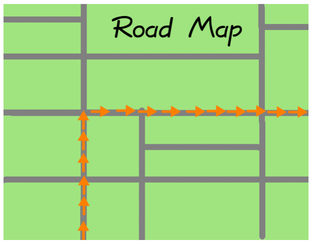
Module 1.6: Working with Functions



When we want to model a financial, scientific, or economic situation, we're usually going to use a function of some sort. Therefore, it is extremely useful for us to practice working with functions. This module assumes that you've encountered the basic idea of a function before—if this is not true, then contact your instructor, who can direct you to some resources.

I have six skills that I'd like to refresh in your memory: evaluating a function; converting words into a function; adding two functions; negating a function; subtracting two functions; multiplying a function by a constant.

The next module will describe two more operations: how to multiply two functions and how to square a function.

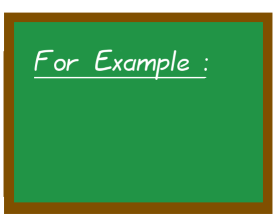
On Page 1212 we're going to see the following function:

$$c(n) = -\frac{n^2}{10,000} + 0.60n + 800$$

that represents the cost of making n guidebooks.

This function concerns a business-estimation analysis about someone who is considering going into the business of printing and selling guidebooks near a famous tourist attraction. The structure of this function might be surprising, but it is true.

We cannot explore the derivation of this function at this time. Even though we don't yet have the tools to construct this function, we should still be skeptical, and see if the function gives us sensible outputs for sensible inputs. This will also give us practice working with this function symbolically.



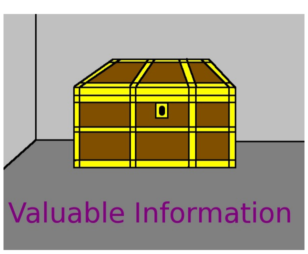
1-6-1

Using the function defined in the previous box, compute the following: What is $c(10)$? What is $c(50)$? What is $c(250)$?

The way you evaluate these is by taking the number in the parentheses (10, 50, or 250, in the above cases) and plugging it in for n in the equation defining the function. However, you must make sure that you plug the number in *everywhere* that you see an n. Since n appears twice, we should be plugging in twice.

When this is done, you will obtain:

$$\begin{aligned} c(10) &= -\tfrac{10^2}{10{,}000} + 0.60(10) + 800 &&= -\tfrac{100}{10{,}000} + 6 + 800 &&= 805.99 \\ c(50) &= -\tfrac{50^2}{10{,}000} + 0.60(50) + 800 &&= -\tfrac{2500}{10{,}000} + 30 + 800 &&= 829.75 \\ c(250) &= -\tfrac{250^2}{10{,}000} + 0.60(250) + 800 &&= -\tfrac{62{,}500}{10{,}000} + 150 + 800 &&= 943.75 \end{aligned}$$

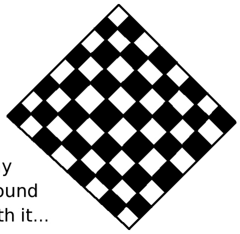


By the way, when we say

$$c(50) = 829.75 \qquad \text{and also} \qquad c(250) = 943.75$$

we call the 50 or the 250 the *input*. Similarly, we call the 829.75 or the 943.75 the *output*.

It is handy to know these vocabulary terms, because they are frequently used.

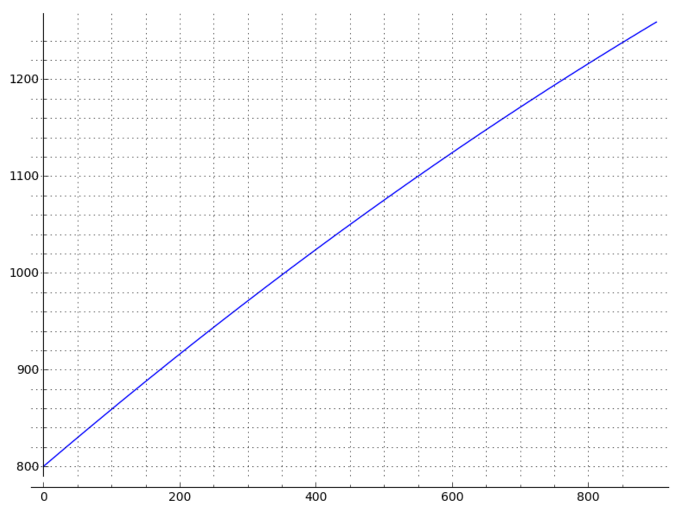


Play
Around
With it...

1-6-2

Let's practice by evaluating $c(n)$ at several other numerical values.

- What is $c(20)$? [Answer: 811.96.]
- What is $c(100)$? [Answer: 859.00.]
- What is $c(500)$? [Answer: 1075.00.]
- Is $c(20)$ equal to double $c(10)$? [Answer: Nope.]
- Is $c(100)$ equal to double $c(50)$? [Answer: Nope.]
- Is $c(500)$ equal to double $c(250)$? [Answer: Nope.]



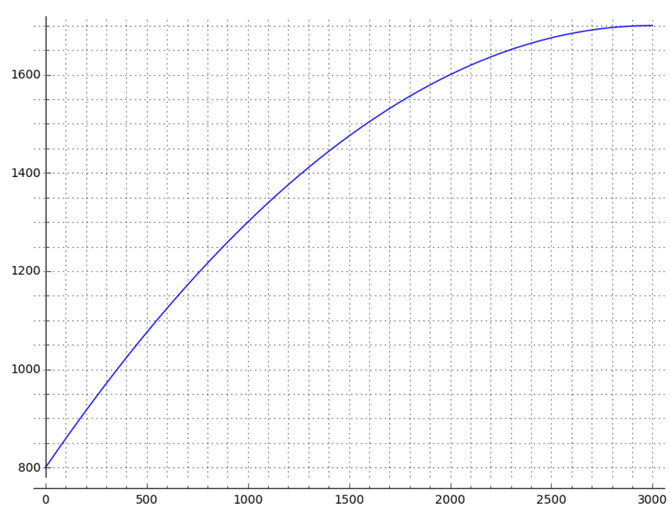
The graph on the left is a plot of $c(n)$ from the last two boxes. As you can see from its equation,

$$c(n) = -\frac{n^2}{10,000} + 0.60n + 800$$

the function $c(n)$ is a second-degree equation, and therefore the graph should be a parabola. Nonetheless, the graph on the left looks like a straight line. How can this be?

First, you can see a bit of curvature in $c(n)$ if you take a piece of paper and use its edge as a ruler. Try to connect the endpoints of $c(n)$ on the graph and you'll see that the shape of $c(n)$ is not quite a straight line.

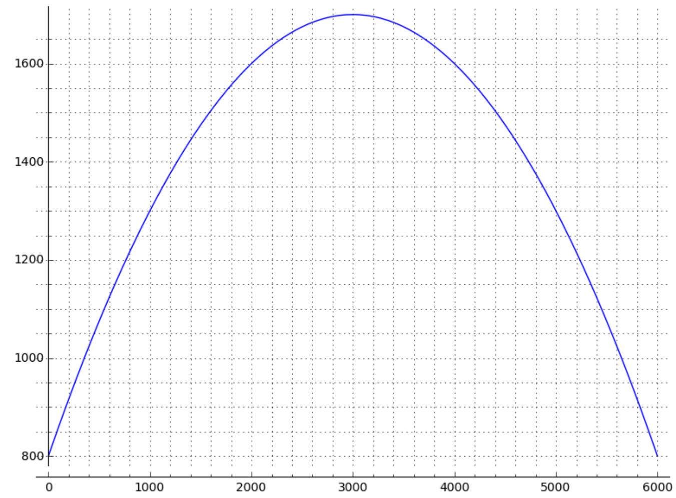
A graph is only a picture, and it tells you very little about a function. It is crucial that you not place very much confidence in graphs at all, as they can mislead you. A fair analogy is that you would not want someone potentially hiring you to judge you solely based on your photograph.



Now I've plotted $c(n)$ again, but over the domain $1 < n < 3000$.

The previous box had a graph that only showed $0 < n < 900$. This partly explains why the graph of $c(n)$ looked like a line. Now it is more clearly curved.

What you're seeing here is called *economy of scale.* The cost per pamphlet is reduced if you produce many of them, because you are carrying out a bulk order. Also, the layout and design of the pamphlet must be paid only once, regardless of how many are made. That's why the function's increase slows down a bit, because of those savings.



The graph to the left is also of $c(n)$, but over the domain $1 < n < 6000$. Finally, it really does look like a parabola

Nonetheless, there are two critical lessons here. First, many students who had access to graphing calculators in high school are accustomed to graphing a function, then looking at the plot, and imagining that they now understand what the function is doing. You can already see how that attitude would lead one astray, because it would make one imagine that $c(n)$ is a line and is always increasing—whereas in truth it is a parabola and not always increasing.

Second, the "U-turn" in this graph is totally unrealistic. This model was intended to be used for values of n roughly below 1000. When you take a function to extremes, far away from its intended range of inputs, then that's called *extrapolation*, and it is a recipe for disaster. We will talk about this some more, later in this textbook.

A simpler example of *extrapolation* would be modeling the performance of a stock over 10 days, and using that information to predict the stock's value 30 years into the future. That's absurd. The model would probably be predictive for a few days into the future, but not farther.

Play Around With it...

1-6-3

Consider $f(x) = 100 + 50\sqrt{x} + 10x$.

- What is $f(16)$? [Answer: $f(16) = 460$.]
- What is $f(25)$? [Answer: $f(25) = 600$.]
- What is $f(36)$? [Answer: $f(36) = 760$.]

For Example :

1-6-4

In the middle of a long problem in a later chapter, we might be modeling the number of subscribers to several websites run by the company you work for. Perhaps your boss estimates that the primary website has 5400 subscribers, and is growing by 100 subscribers per week. A secondary website has 2300 subscribers, but is growing by 300 subscribers per week. A new website has only 300 subscribers, but is growing by 100 subscribers per week. What function $p(w)$ would describe the number of subscribers for the primary site, predicting w weeks into the future? (Assume that your boss's estimates are good ones.)

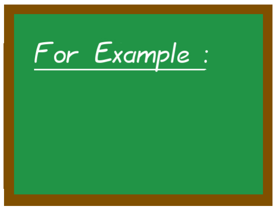
$$p(w) = 5400 + 100w$$

Play Around With it...

1-6-5

Continuing with the previous box...

- What function $s(w)$ would describe the number of subscribers at the secondary website? [Answer: $s(w) = 2300 + 300w$.]
- What function $n(w)$ would describe the number of subscribers at the new website? [Answer: $n(w) = 300 + 100w$.]



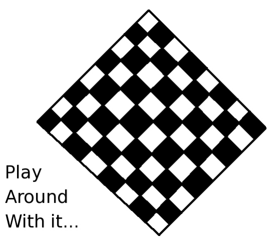
1-6-6

Vito has recently opened a big-screen TV wholesale store in The Bronx. Electronics stores can contact him and request deliveries of as many TVs as they'd like. Meanwhile, Vito's associates can drop off any big-screen TVs that they "acquire," and Vito won't ask them any embarrassing questions. The deliveries are accomplished by a company just down that street that has an old but spacious moving van. The van is large enough that it is very unlikely that there would be an order too large for it to carry. We're going to write some functions down that will model Vito's business.

First, there is a transportation cost. The delivery company charges Vito $ 75 per trip, plus $ 0.15 per pound transported. Second, there is a packaging cost. Each TV requires $ 17.50 of packing and wrapping material, and $ 5 of packing labor (representing 20 minutes of time from his nephew Vinny, who makes $ 15/hour). Each TV, once packed and wrapped, weighs 73 lbs. (We're making a simplifying assumption that all the TVs are the same model.)

Let's start with the packing cost. If an order has x TVs, then the packing cost is

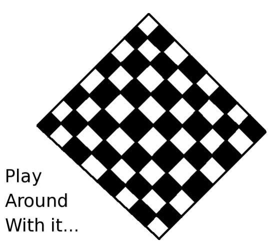
$$p(x) = 17.50x + 5x = 22.50x$$



1-6-7

Using the data in the previous box...

- Tell me a function $w(x)$ that computes the weight of a shipment of x TVs. [Answer: $w(x) = 73x$ pounds.]
- Tell me a function $s(x)$ that computes the shipping cost of a shipment of x TVs. [Answer: $s(x) = 75 + 0.15(73x)$.]
- Write the previous function without parentheses. [Answer: $s(x) = 75 + 10.95x$.]



1-6-8

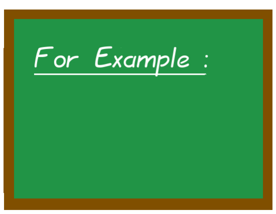
The best way to check your work in a situation like this is to compute what you think should be the answer for two situations (maybe $x = 3$ and $x = 5$) using only your business sense—no algebra, just a calculator and your business savvy. Then, see if the function gives you the same outputs for those inputs.

- What should the weight of 3 TVs actually be? [Answer: 219 lbs.]
- What should the shipping cost for 3 TVs actually be? [Answer: $ 107.85.]
- What should the packaging cost for 3 TVs actually be? [Answer: $ 67.50.]
- What should the weight of 5 TVs actually be? [Answer: 365 lbs.]
- What should the shipping cost for 5 TVs actually be? [Answer: $ 129.75.]
- What should the packaging cost for 5 TVs actually be? [Answer: $ 112.50.]

Okay, now let's take the inputs $x = 3$ and $x = 5$, and plug them into our functions. Let's see if we get the same numbers as in the previous box for our outputs.

$$\begin{aligned} w(3) &= 73(3) = 219 \quad \checkmark \\ s(3) &= 75 + 10.95(3) = 75 + 32.85 = 107.85 \quad \checkmark \\ p(3) &= 22.50(3) = 67.50 \quad \checkmark \\ w(5) &= 73(5) = 365 \quad \checkmark \\ s(5) &= 75 + 10.95(5) = 75 + 54.75 = 129.75 \quad \checkmark \\ p(5) &= 22.50(5) = 112.50 \quad \checkmark \end{aligned}$$

It is a very happy day when things work out nicely. In this case, they worked out absolutely exactly. Therefore, we can be confident that we found the correct functions.



1-6-9

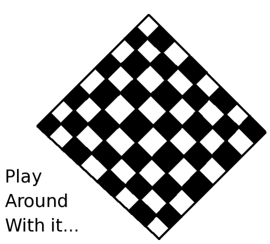
Now we're going to see how adding two functions works symbolically, before we try it in a word problem. Consider $a(x) = 17 - 3x$ and $b(x) = 8 + 5x$. Then

$$\begin{aligned} a(x) + b(x) &= (17 - 3x) + (8 + 5x) \\ &= 17 + 8 - 3x + 5x \\ &= 25 + 2x \end{aligned}$$

As you can see, when we add together two functions, we take care to combine like terms in order to produce a simpler function.

Likewise, if $f(x) = 82 + 3x - x^3$ and $g(x) = 100 - 3x + x^2$, then

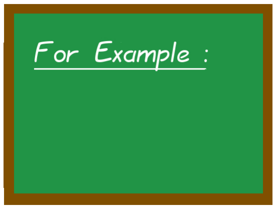
$$\begin{aligned} f(x) + g(x) &= (82 + 3x - x^3) + (100 - 3x + x^2) \\ &= 82 + 100 + 3x - 3x + x^2 - x^3 \\ &= 182 + x^2 - x^3 \end{aligned}$$



1-6-10

Let's practice this skill of adding functions momentarily, before moving on.

- If $c(x) = 103 - 15x$ and $d(x) = 17 + 15x + x^2$, then what is $c(x) + d(x)$?
  [Answer: $c(x) + d(x) = 120 + x^2$.]
- If $r(x) = 88 + x^2$ and $s(x) = x^3 + x^4$, then what is $r(x) + s(x)$?
  [Answer: $r(x) + s(x) = 88 + x^2 + x^3 + x^4$.]
- If $h(x) = 5x - 18$ and $k(x) = 9 - 3x$, then what is $h(x) + k(x)$?
  [Answer: $h(x) + k(x) = 2x - 9$.]



1-6-11

Let's think back to the problem about Vito's wholesale store in The Bronx, selling TVs. It was on Page 170 of this module. We had three functions:

$$\begin{aligned} p(x) &= 22.50x \text{ (packaging cost)} \\ w(x) &= 73x \text{ (weight)} \\ s(x) &= 75 + 10.95x \text{ (shipping cost)} \end{aligned}$$

Let's compute a function $c(x)$ that represents the total cost (shipping and packaging) for an order with x TVs. Clearly, we have to add the packaging and shipping functions.

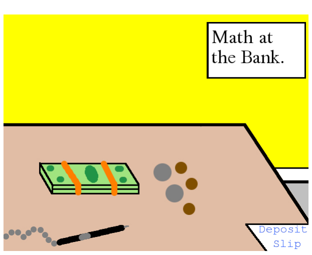
$$\begin{aligned} c(x) &= p(x) + s(x) \\ &= 22.50x + 75 + 10.95x \\ &= 75 + (22.50 + 10.95)x \\ &= 75 + 33.45x \end{aligned}$$

Notice, we did not add the weight function.

Here is a table which summarizes the entire situation with Vito's wholesale TV business.

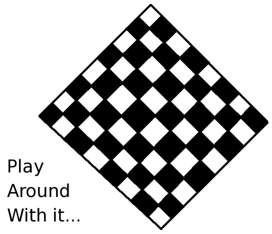
Number of TVs	Weight		Shipping Cost	Packaging Materials Cost	Packaging Wages	Packaging Sub-total Cost	Grand Total Cost
1	73	lbs	\$ 85.95	\$ 17.50	\$ 5.00	\$ 22.50	\$ 108.45
2	146	lbs	\$ 96.90	\$ 35.00	\$ 10.00	\$ 45.00	\$ 141.90
3	219	lbs	\$ 107.85	\$ 52.50	\$ 15.00	\$ 67.50	\$ 175.35
4	292	lbs	\$ 118.80	\$ 70.00	\$ 20.00	\$ 90.00	\$ 208.80
5	365	lbs	\$ 129.75	\$ 87.50	\$ 25.00	\$ 112.50	\$ 242.25
6	438	lbs	\$ 140.70	\$ 105.00	\$ 30.00	\$ 135.00	\$ 275.70
7	511	lbs	\$ 151.65	\$ 122.50	\$ 35.00	\$ 157.50	\$ 309.15
8	584	lbs	\$ 162.60	\$ 140.00	\$ 40.00	\$ 180.00	\$ 342.60
9	657	lbs	\$ 173.55	\$ 157.50	\$ 45.00	\$ 202.50	\$ 376.05
10	730	lbs	\$ 184.50	\$ 175.00	\$ 50.00	\$ 225.00	\$ 409.50
11	803	lbs	\$ 195.45	\$ 192.50	\$ 55.00	\$ 247.50	\$ 442.95
12	876	lbs	\$ 206.40	\$ 210.00	\$ 60.00	\$ 270.00	\$ 476.40
13	949	lbs	\$ 217.35	\$ 227.50	\$ 65.00	\$ 292.50	\$ 509.85
14	1022	lbs	\$ 228.30	\$ 245.00	\$ 70.00	\$ 315.00	\$ 543.30
15	1095	lbs	\$ 239.25	\$ 262.50	\$ 75.00	\$ 337.50	\$ 576.75
x	$w(x)$		$s(x)$			$p(x)$	$c(x)$

It is worth it to pick some input, like $x = 7$, and verify that the table agrees with you. Verify this using a calculator and business sense, not algebra. Then verify, for that same input, that the functions $s(x)$, $p(x)$, and $c(x)$, produce outputs that match both the table and yourself. Then you'll be confident that we've done everything correctly.



By the way, the table above is something that should be easily constructed in MS-Excel by any business student, even a freshman. It took me about 8 minutes, so you should be able to do it in 4–16 minutes.

Proficiency in MS-Excel is probably the most important skill of all for any business student. If you are not skilled in MS-Excel, then you should try to remedy that at a convenient time, such as during a break between two semesters, or by taking a course.



1-6-12

Continuing with the previous sequence of boxes, let's imagine that Vito has a close call with the police from the local precinct. They are very forgiving and cooperative with him. Therefore, he decides that it is time to provide them with a little "supplemental income" consisting of $ 50 per delivery, plus $ 20 per TV. (It is very expensive to elevate yourself above the law.)

Revise your function for the total cost (shipping, packaging, and bribing) for an order with x TVs. [Answer: $c(x) = 125 + 53.45x$.]

What we did above suggests a method for checking our work. While you would never want to construct a large table (like in the previous box) only to check your work, you can check the sum of two functions by just picking a number. Plug that number into both of the original functions, as well as your newly produced function. Then, check to see if the addition comes out correct. This is one of those concepts that is easier to demonstrate than to describe, so let's see how it is done.

In the checkerboard back on Page 171 we computed

$$c(x) + d(x) = 120 + x^2$$

where the original functions were

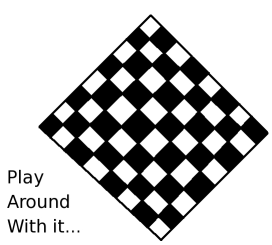
$$c(x) = 103 - 15x \qquad \text{and} \qquad d(x) = 17 + 15x + x^2$$

We can compute $c(42) = -527$ and $d(42) = 2411$. Then when we plug 42 into $120 + x^2$, we obtain 1884. This is good, because

$$-527 + 2411 = 1884$$

so we are confident that we properly combined $c(x)$ and $d(x)$ into one function.

Generally, I don't plug in 42, because that can get messy. My favorites to plug in are 0, 1, and 2—because they are smaller and easier to use.

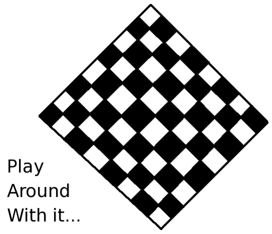


1-6-13

Let's think back to the problem about modeling the number of subscribers for three websites, on Page 169 of this module. We had three functions:

$$\begin{aligned} p(w) &= 5400 + 100w \text{ (primary website)} \\ s(w) &= 2300 + 300w \text{ (secondary website)} \\ n(w) &= 300 + 100w \text{ (new website)} \end{aligned}$$

Now write a function $t(w)$ that represents the total number of subscribers among all three websites. [Answer: $t(w) = 8000 + 500w$.]



1-6-14

Let's consider a business that is selling bottles of wine. They have a range of varieties, from the cheap to the expensive, but since they are all in 750 mL bottles, they each weigh the same amount. The type of box used to pack the bottles can hold up to 24 bottles and costs \$ 4.75, plus 60 cents of bubble wrap per bottle. For shipping costs, the postage is \$ 19.95 plus \$ 8.82 per bottle. Let n be the number of bottles ordered, from 1 to 24.

- What is the shipping cost function, $s(n)$? [Answer: $s(n) = 19.95 + 8.82n$.]
- What is the packing cost function, $p(n)$? [Answer: $p(n) = 4.75 + 0.60n$.]
- What function $h(n)$ would represent the total for shipping & packing? [Answer: $h(n) = 24.70 + 9.42n$.]

Our functions in the previous box only make sense if the order is for 1–24 bottles of wine. If the order were for 0 bottles, then we have no shipping expenses. At 25 bottles, we need a second box, which will throw everything off. We can write this mathematically in the following way:

$$\begin{aligned} s(n) &= 19.95 + 8.82n \text{ where} 1 \leq n \leq 24 \\ p(n) &= 4.75 + 0.60n \text{ where} 1 \leq n \leq 24 \\ h(n) &= 24.70 + 9.42n \text{ where} 1 \leq n \leq 24 \end{aligned}$$

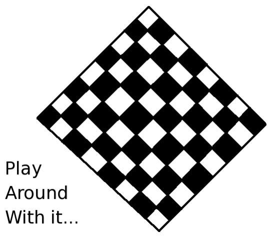
The symbols $1 \leq n \leq 24$ just mean that n is between 1 and 24.

In the previous problem (about shipping wine) the limitation of "1 to 24" was not really very important, because it was obvious in the context of the problem. However, when we study Break-Even Analysis on Page 360, and the theory of Supply & Demand on Page 435, we will see cases where extremely strange things can happen when you stray out of the *domain* of numbers for which a particular function makes sense. There, the domain is not obvious.

The word *domain* refers to the set of numbers over which a function has meaning and can be computed. In the preceding box we reasoned why the positive integers between (and including) 1 to 24 belong in the domain of the function. Contrastingly, the negative integers, zero, and integers greater than 24 are not in the function's domain.

In pure mathematics it can be an interesting problem to compute the domain of a function , but in applied mathematics the domain really depends on the particular problem being solved and its context.

Another example of a domain violation is in the opening example of this module. There, we saw $c(n)$, which made sense for small values of n. Yet, for large values of n, such as 3000 and above, the function made no sense at all. It predicted that it would be cheaper to buy $n = 5000$ pamphlets than $n = 4000$ pamphlets, which is clearly not true!



1-6-15

Bob has an internship with a realtor and is making nice brochures for each house that the realtor has been hired to sell. The brochures are made by printing on both sides of a standard piece of card-stock the size of an ordinary piece of paper, and the page is then folded up to make the brochure. The card-stock is cheap but the problem is that color inkjet cartridges are very expensive, and furthermore, real-estate brochures are almost entirely pictures and so they use up a lot of ink. One has to keep the costs under control. Luckily, there are websites that track this sort of thing. Bob has four printers under consideration.

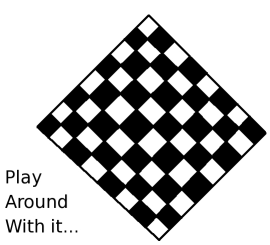
1. Printer A costs \$ 59, and a page of photographs is quoted as costing 89 cents.
2. Printer B costs \$ 169, and a page of photographs is quoted as costing 41 cents.
3. Printer C costs \$ 499, and a page of photographs is quoted as costing 33 cents.
4. Printer D costs \$ 3999, and a page of photographs is quoted as costing 19 cents.
5. As an alternative, the local copy center down the road charges 59 cents per full-color page.

Now write down a function that describes how much it would cost to buy each printer and print n brochures on it.

- What is the function $a(n)$ for Printer A? [Answer: $a(n) = 59 + 1.78n$.]
- What is the function $b(n)$ for Printer B? [Answer: $b(n) = 169 + 0.82n$.]
- What is the function $c(n)$ for Printer C? [Answer: $c(n) = 499 + 0.66n$.]
- What is the function $d(n)$ for Printer D? [Answer: $d(n) = 3999 + 0.38n$.]
- What is the function $\ell(n)$ if Bob just uses the local copy center down the road? [Answer: $\ell(n) = 1.18n$.]

If the previous checkerboard problem confused you, then ask yourself, did you read the words of the problem very carefully? Some students do, and some students skip the words and only read the numbers. It is key to realize that we are printing on *both* sides of the card stock, so making 200 brochures is equivalent to printing 400 pages of photographs.

Always read the problem carefully!



1-6-16

Now we can analyze which printer should be purchased. We are going to have to do this a bit slowly, but on Page 798 we'll learn a technique that will allow us to compute more rapidly. For now, Bob's boss has friends in several towns in the area. First, there is Fred who lives in a medium-sized town, and he imagines that he'll print 600 brochures over the lifespan of the printer.

- What would Fred's cost be for each printer?

  [Answer: $a(600) = \$\ 1127$; $b(600) = \$\ 661$; $c(600) = \$\ 895$;

  $d(600) = \$\ 4227$; $\ell(600) = \$\ 708$.]
- Which printer should Fred buy? [Answer: He should choose Printer B.]

Play Around With it...

1-6-17

Second, there is Bubba, who lives in a rural small town, where he is the only realtor and so he has no competition. Therefore, he imagines that he'll print 200 brochures over the lifespan of whatever printer he buys.

- What would Bubba's cost be for each printer?

  [Answer: $a(200) = \$\ 415$; $b(200) = \$\ 333$; $c(200) = \$\ 631$;

  $d(200) = \$\ 4075$; $\ell(200) = \$\ 236$.]
- Which printer should Bubba buy? [Answer: He should use the local copy center.]

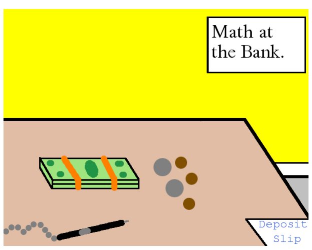
Play Around With it...

1-6-18

Third, there is Aaron, who lives in New York City and has a staff of twenty. He imagines that he'll print 9000 brochures over the lifespan of his printer.

- What would Aaron's cost be for each printer?

  [Answer: $a(9000) = \$\ 16{,}079$; $b(9000) = \$\ 7549$; $c(9000) = \$\ 6439$; $d(9000) = \$\ 7419$; $\ell(9000) = \$\ 10{,}620$.]
- Which printer should Aaron buy? [Answer: He should buy Printer C.]



The previous three examples demonstrate an important concept in business: different types of consumers (businesses, individuals, large corporations, and so forth) have different needs. If this were not the case, then one particular printer model would satisfy the needs of all customers in the market. As a result, all other printers would be redundant, no one would buy them, and their manufacturers would go out of business (or their manufacturers would shut down those specific product lines and make other things.)

Therefore, companies have to be careful to make different lines of products for different types of consumers, each filling a specialized niche in the market.

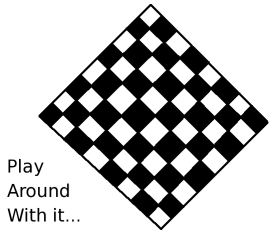
In any case, each business must make a careful choice to figure out which product will fulfill the customers' needs at the minimum cost. When we study Cost-Benefit Analysis on Page 386, we will learn how to handle these situations more precisely (and much more quickly).

Play Around With it...

1-6-19

Frank's company makes small medical electronic devices, and they've just opened a branch in Australia. In addition to dealing with the time difference and minor cultural differences, they have to figure out how they'll communicate with their new clients and offices overseas. There are three options: calling on a mobile phone, calling on a land-line, and using a calling card. It turns out that making international calls on a mobile phone is phenomenally expensive, so that's ruled out. Using a land-line, it costs 15 cents per minute, plus a ten cent "connection fee" per call.

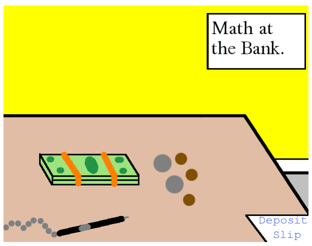
- What function $\ell(t)$ represents the cost of making a land-line call of duration t minutes? [Answer: $\ell(t) = 0.10 + 0.15t$.]
- How much would a three-minute phone call cost, on a land-line? [Answer: $\ell(3) = 0.55$, or 55 cents.]
- How much would a one-hour phone call cost, on a land-line? [Answer: $\ell(60) = 9.1$, or \$ 9.10.]



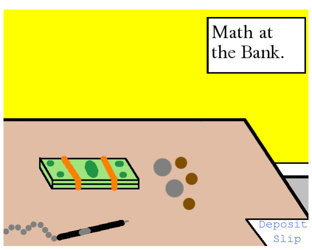
1-6-20

Continuing with the previous box, using a prepaid calling card it would cost Frank 3 cents per minute to call Australia, but the connection fee is 75 cents.

- What function $c(t)$ represents the cost of making a calling-card call of duration t minutes?
  [Answer: $c(t) = 0.75 + 0.03t$.]
- How much would a three-minute phone call cost, with a calling card?
  [Answer: $c(3) = 0.84$ or 84 cents.]
- How much would a one-hour phone call cost, with a calling card?
  [Answer: $c(60) = 2.55$ or $ 2.55.]



This type of situation is not uncommon. If Frank thinks that most phone calls are going to be around three minutes, then the land-line is cheaper (55 cents versus 84 cents). However, if Frank thinks that most phone calls are going to be around an hour, then the calling card is cheaper ($ 2.55 versus $ 9.10).



There is a technique for turning data into functions, called *regression*, which you might have seen before, or perhaps not. One often generates a linear or quadratic function that best represents the data (where "best" is a word that we will qualify and precisely define later). Sometimes the linear function is called a *best-fit line*, a *trend line*, or a *line of best fit*. We will study this concept a lot, throughout this textbook.

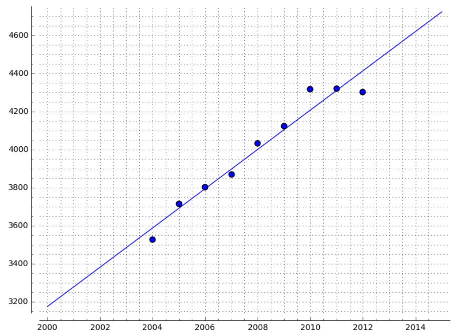
In finance, economics, and the sciences, regressions are an extremely popular tool because often one is not in possession of a symbolic function to model a data set. However, one can collect data, analyze it, and then use software such as a spreadsheet program or a computer algebra package to determine the regression.

We're now going to see how this works by analyzing some enrollment data at a university.

The following data represents undergraduate enrollment at the University of Wisconsin—Stout campus (Wisconsin's Polytechnic University), where I am currently teaching. The data comes from the "UW Stout 2012–2013 Fact Book" and previous editions of the same.

	2004	2005	2006	2007	2008	2009	2010	2011	2012
males	3526	3714	3802	3868	4032	4123	4318	4319	4300
females	3447	3627	3713	3683	3737	3848	3985	4034	3970

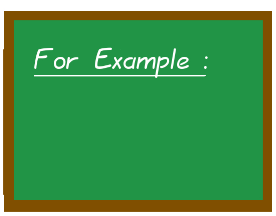
We will work with this data in the next three boxes.



male data

female data

In this box you can see the plots of the enrollment data together with their best-fit lines or linear regressions. As you can see, the data points lie fairly close to the line, and the error in almost all cases (by which we mean the distance between the true dot and the prediction or line) is less than 25 students.



1-6-21

I have not yet taught you how to compute the regression of data into a best-fit line. However, it turns out that the best fit lines for the data set in the previous box are

$$m(t) = (103.300\cdots)t + 3173.82\cdots$$

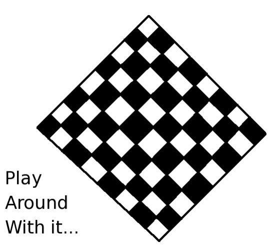
as well as

$$f(t) = (67.0333\cdots)t + 3246.39\cdots$$

where $t = 0$ represents the year 2000. (That's a frequently used way to handle t, by the way.) We will learn to compute those regression functions ourselves, later. For now, what should we do if we want a function for predicting total enrollment?

Armed with this information, if we want to know the total enrollment $e(t)$, we would just add the functions:

$$\begin{aligned} e(t) &= m(t) + f(t) \\ &= (103.300\cdots)t + 3173.82\cdots + (67.0333\cdots)t + 3246.39\cdots \\ &= (170.3333\cdots)t + 6420.21\cdots \end{aligned}$$



1-6-22

First we're going to test the functions in the previous two boxes, namely $m(t)$, $f(t)$, and $e(t)$, to see how they predict enrollment for years which have hard data. We will use 2008 as our test case.

- What does $f(t)$ predict as the female enrollment for 2008? [Answer: $3782.65\cdots$.]
- What does $m(t)$ predict as the male enrollment for 2008? [Answer: $4000.22\cdots$.]
- What does $e(t)$ predict as the total enrollment for 2008? [Answer: $7782.87\cdots$.]

Play Around With it...

1-6-23

Second, let's quantify how accurate those 2008 results are, compared to the actual enrollment figures we have. We will study this idea in more detail later (on Page 305), but the formula is

$$\text{relative error} = \frac{\text{prediction} - \text{truth}}{\text{truth}}$$

Using that formula...

- ...what is the relative error in the prediction of female enrollment for 2008? [Answer: $0.0122156\cdots$, or $+1.22156\%$.]
- ...what is the relative error in the prediction of male enrollment for 2008? [Answer: $-0.00788194\cdots$, or -0.788194%.]
- ...what is the relative error in the prediction of total enrollment for 2008? [Answer: $0.00178530\cdots$, or $+0.178530\%$.]

Most models you will encounter in this course and in later courses have much higher error rates. This is a surprisingly accurate model. (Also, in industry, we tend to report percentages to nearest 1% of 1%. That means we'd write +1.22% in place of +1.22156%. However, we are using six significant figures in this textbook.)

Play Around With it...

1-6-24

Now that we're relatively confident with the quality of these functions that came from a regression, let's use them to make an actual prediction for 2015.

- What does $m(t)$ predict as the male enrollment for 2015? [Answer: 4723.32.]
- What does $f(t)$ predict as the female enrollment for 2015? [Answer: $4251.88\cdots$.]
- What does $e(t)$ predict as the total enrollment for 2015? [Answer: $8975.20\cdots$.]

I'm sure you can see that being able to make predictions of this sort, even though they are not guaranteed to be accurate, can be a highly desirable skill in business administration, government, not-for-profits, the sciences, and even academic administration.

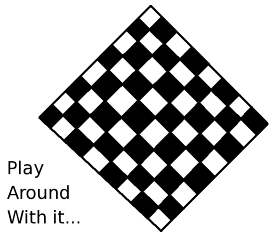
Play Around With it...

1-6-25

Now that we're relatively confident with the quality of these functions that came from a regression, let's use them to make an actual prediction for 2016.

- What does $m(t)$ predict as the male enrollment for 2016? [Answer: 4826.62.]
- What does $f(t)$ predict as the female enrollment for 2016? [Answer: $4318.92\cdots$.]
- What does $e(t)$ predict as the total enrollment for 2016? [Answer: $9145.54\cdots$.]

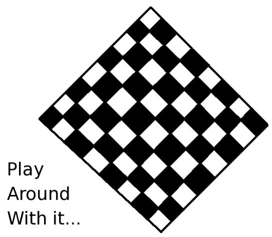
I'm sure you can see that being able to make predictions of this sort, even though they are not guaranteed to be accurate, can be a highly desirable skill in business administration, government, not-for-profits, the sciences, and even academic administration.



1-6-26

As you can tell, I wrote the last few boxes back in 2013 or so. I wrote this box in 2017, which means that we can compare our predictions with the truth for Fall 2015 and Fall 2016. As it turns out, for 2015, there were 8388 students total: 4554 males and 3834 females.

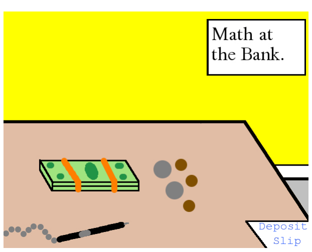
- What was our relative error for males in 2015? [Answer: $0.0371805\cdots$ or $+3.71805\%$.]
- What was our relative error for females in 2015? [Answer: $0.108993\cdots$ or $+10.8993\%$.]
- What was our relative error for students in 2015? [Answer: $0.0700047\cdots$ or $+7.00047\%$.]



1-6-27

Continuing with the previous box, we can now analyze the performance of our model for 2016. In 2016, there were 8398 students total: 4594 males and 3804 females. Let's see how we did.

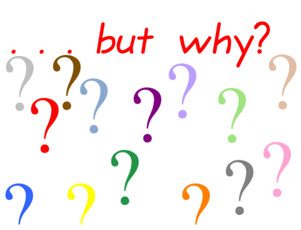
- What was our relative error for males in 2016? [Answer: $0.0506356\cdots$ or $+5.06356\%$.]
- What was our relative error for females in 2016? [Answer: $0.135362\cdots$ or $+13.5362\%$.]
- What was our relative error for students in 2016? [Answer: $0.0890140\cdots$ or $+8.90140\%$.]



Predicting the future is difficult. Accordingly, if you can get a relative error of $\pm 10\%$ or less, then you've done well. As you can see from the previous sequence of boxes, our relative errors were not horrible when it comes to predicting the future. Yet, they also were far from as low as they were when we were working with known data. This almost always happens.

Whether you call it a linear regression, a best-fit line, a trend line, or a line of best fit, this tool is fairly good and making short-term predictions of the future. Therefore, it is a staple and a mainstay of the modern business world.

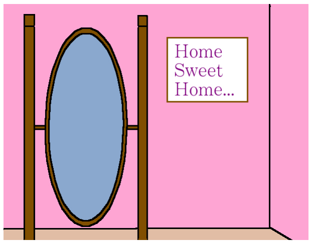
For completeness, I should note that while we were able to do "okay" for predicting 3–4 years after our data set had ended, it would be foolhardy to predict 10, 15, or 20 years out. That would be *extrapolation*, first explained on Page 169, which is a recipe for disaster.



We've seen a whole bunch of examples of operations on functions now, and I'd like to recap by re-examining how the addition of functions might be useable (or not useable) in those examples.

With the two shipping examples (the TVs and the bottles of wine), adding the functions made sense because we were combining two types of costs—packing materials and wages, or postage and packing materials. Likewise, when working with the models tabulating the number of male and female students, adding made sense because we were combining two categories of students—male and female.

It is not always useful, however, to add functions together, even if they relate to the same problem. Adding the functions in the brochure example would be like printing each brochure five times, once on each printer and also at the copy center—but that doesn't make any sense. Likewise, in the international-calling plans example, you either pay your phone's service provider or you pay the calling-card company—you would not pay both for the same call, so adding the cost functions would not make sense.

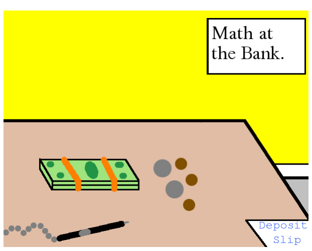


A Pause for Reflection...
I'd like to pause now with a detour into some philosophical matters; we'll return to mathematics shortly.

Data collection is always a powerful tool, but it becomes more powerful when performed over a series of years. Enrollment data has probably been tracked since UW-Stout's founding, in 1891. However, the statistical community doesn't look at the world the same way in 2013 as it did in 1891. For example, our thoughts about gender are much more nuanced than they once were.

UW Stout—at the time called " the Stout Manual Training School"—created a bit of scandal by admitting female students from the very first day, in an era when many colleges prohibited female students. By comparison, Fordham University, where I have also taught, did not permit any female students until eighty-three years later, in 1974—and it was founded fifty years prior to Stout!

In the past two decades, in stark contrast with the 1890s, the universities of the world have come to understand that sometimes students will change their gender. I know of at least one very recent graduate who started the undergraduate career male and who graduated female. While such transformations are rare, it is probably a good idea for surveys to have a box for "transgendered" alongside the traditional "male" and "female" options. However, one cannot recompute the historical data that way, so sometimes it is unfortunately necessary to keep track of data using older, outdated ways of thinking.



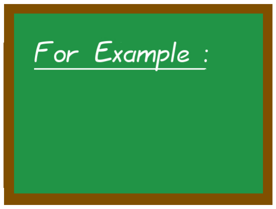
A really important example of collecting data the wrong way deliberately, to maintain comparisons with older data sets, would be the Unemployment Rate. The official Unemployment Rate is tabulated by the Bureau of Labor Statistics monthly (with weekly updates) and it is an extraordinarily important number, not only for the economy, but also for the re-election of politicians.

In any case, it became very clear in the early 1990s that the official Unemployment Rate was badly handling several situations, the most common of which was underemployment. Consider someone with a professional or scientific background who loses their job, and takes part-time employment to have money with which to buy food and to avoid becoming homeless. The official Unemployment Rate would count this person as "employed" and therefore economically satisfied. However, few of us would view this highly educated person as satisfied.

To accommodate this and other considerations, new methods of calculation were adopted in 1994, and now there are six different unemployment rates. The official Unemployment Rate is U-3, but there is also U-6, which accounts for the situation we described above as an "underemployed" and therefore economically dissatisfied worker.

However, because the U-6 was only created in 1994, there is not enough historical data to foster data between the present day and previous booms or recessions. For example, in 2014, one cannot compare the recession of 2008–2012 to previous recessions, because there's only one other recession in the U-6 data set. For this reason, economists and commentators must keep referring to the U-3, despite all of its flaws, since that is the only data set that goes back to 1947, whereas the other auxiliary data sets go back to the 1990s.

Okay, we've now practiced evaluating functions, turning words into functions, and adding functions. We are exactly halfway through our objectives. Now we'll look at negating a function, subtracting functions, and multiplying functions by a constant.



Flipping a number from positive to negative or negative to positive is an extremely easy operation—you just change the sign in front of the number. Likewise, it is also an easy operation to perform on a function, except that you must flip the sign on each term. Consider how to compute the negation of $f(x) = 17 + 3x - x^2$.

We need to flip the sign of each term. The $+17$ becomes -17, the $+3x$ becomes $-3x$, and the $-x^2$ becomes $+x^2$. We then have $-f(x) = -17 - 3x + x^2$.

1-6-28

When you flip the sign of a number and add it to the original, you get zero. This is also true with functions. Observe, the sum of our function and its negation total to zero.

$$(17 + 3x - x^2) + (-17 - 3x + x^2) = 17 - 17 + 3x - 3x - x^2 + x^2 = 0 + 0 - 0 = 0$$

However, it is faster to use my favorite trick: that of plugging in 0, 1, and 2. If corresponding results are opposites of each other, then you know that you have correctly found the function's inverse.

$$\begin{array}{lclllcll} 17 + 3(0) - (0)^2 & = & 17 & \text{while} & -17 - 3(0) + (0)^2 & = & -17 & \text{(Good!)} \\ 17 + 3(1) - (1)^2 & = & 19 & \text{while} & -17 - 3(1) + (1)^2 & = & -19 & \text{(Good!)} \\ 17 + 3(2) - (2)^2 & = & 19 & \text{while} & -17 - 3(2) + (2)^2 & = & -19 & \text{(Good!)} \end{array}$$

The operation of negation seems so trivial, that you might be surprised that I'm teaching it. Let's look one last time at negating the function

$$f(x) = 17 + 3x - x^2$$

which we were doing in the last two boxes. The risk is that instead of flipping all the terms

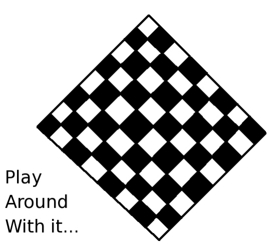
$$-f(x) = -17 - 3x + x^2 \qquad \leftarrow \text{CORRECT!}$$

some students will flip only the first term

$$-f(x) = -17 + 3x - x^2 \qquad \leftarrow \text{WRONG!}$$

which is incorrect. This is an extremely common mistake.

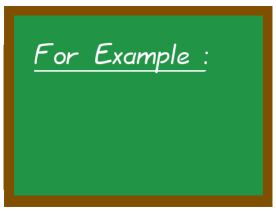
For example, plugging in $x = 1$ to the incorrect version of $-f(x)$ would yield -15 (a wrong answer) instead of -19 (the right answer). Just because you put a minus sign in front of the first term of a function does not mean that you have negated the function—you must distribute that minus sign to all terms within the function.



Compute the negations of the following two functions.

- What is the negation of $f(x) = 9 - x^2 + 3x$? [Answer: $-f(x) = -9 + x^2 - 3x$.]
- What is the negation of $g(x) = 5 - x + x^3$? [Answer: $-g(x) = -5 + x - x^3$.]

1-6-29



1-6-30

Let's consider that you have two functions,

$$f(x) = x^3 - x^2 + 3x + 8 \text{ and } g(x) = x^3 + x^2 - 9x + 23$$

and that you are asked to compute $h(x) = f(x) - g(x)$. Your computation should look like this:

$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= \left(x^3 - x^2 + 3x + 8\right) - \left(x^3 + x^2 - 9x + 23\right) \\ &= x^3 - x^2 + 3x + 8 - x^3 - x^2 + 9x - 23 \\ &\quad \text{Notice: we just negated } g(x)! \\ &= x^3 - x^3 - x^2 - x^2 + 3x + 9x + 8 - 23 \\ &= -2x^2 + 12x - 15 \end{aligned}$$

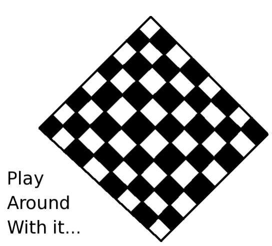
We can conclude that our final answer is

$$h(x) = -2x^2 + 12x - 15$$

I'd like to point out that the critical step is where we negate $g(x)$. We have to make sure to flip all the signs on $g(x)$. Many students will only flip the first one, and that's where I see most mistakes. In fact, the only reason we teach the negation of a function is to help students be able to subtract functions correctly.

Just like with adding functions, we can check our work by plugging in the values $x = 0$, $x = 1$, and $x = 2$, and seeing if the arithmetic for each value works out as expected.

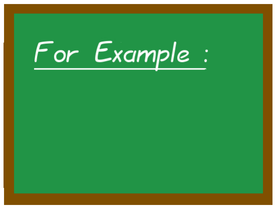
- Using $x = 0$ we see that $f(0) = 8$, $g(0) = 23$, and $h(0) = -15$, which is good because $8 - 23 = -15$.
- Using $x = 1$ we see that $f(1) = 11$, $g(1) = 16$, and $h(1) = -5$, which is good because $11 - 16 = -5$.
- Using $x = 2$ we see that $f(2) = 18$, $g(2) = 17$, and $h(2) = 1$, which is good because $18 - 17 = 1$.



1-6-31

Now let's practice this algebraic skill before we use it to solve some word problems.

- If $a(x) = 12 - x^2$ and $b(x) = 3x + 12$, then what is $a(x) - b(x)$?
  [Answer: $a(x) - b(x) = -x^2 - 3x$.]
- If $f(x) = 3x + 5$ and $g(x) = x^2 + 3x$, then what is $g(x) - f(x)$?
  [Answer: $g(x) - f(x) = x^2 - 5$.]
- If $r(x) = 4 - 9x + x^2$ and $s(x) = x^2 - 9x + 4$, then what is $r(x) - s(x)$?
  [Answer: $r(x) - s(x) = 0$.]
- If you got the first three correct, you may skip this fourth one.
- If $p(x) = x^2 + 4x + 8$ and $q(x) = -x^2 - 4x + 8$ then what is $p(x) - q(x)$?
  [Answer: $p(x) - q(x) = 2x^2 + 8x$.]



1-6-32

Suppose that the marketing club at your university wants to raise money by selling coffee mugs with a cool and humorous logo. There is a website which specializes in such things, and it will cost \$ 4 per mug, plus \$ 150 to create the design. The club votes to sell the mugs for \$ 9 each. Now let's create a cost function, a revenue function, and a profit function, for this business venture.

The cost of producing n mugs would be

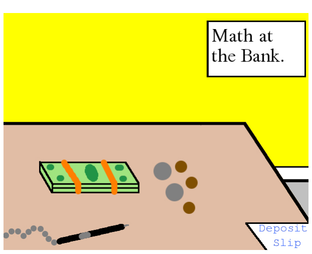
$$c(n) = 150 + 4n$$

because the club must pay the \$ 150 up front, and each mug costs \$ 4. The revenue function would be even simpler, namely $r(n) = 9n$.

We will compute the profit function in the next box.

Continuing with the previous box, the profit function is the revenue function minus the cost function. This is an excellent example of subtracting functions. In this case we have

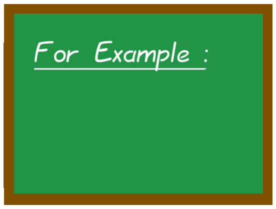
$$\begin{aligned} p(n) &= r(n) - c(n) \\ &= 9n - (150 + 4n) \\ &= 9n - 150 - 4n \\ &= 5n - 150 \end{aligned}$$



We can summarize the situation in the previous box with the following table. The three functions we created—revenue, cost, and profit—were used to generate the data in the middle three columns. You can just pick a few rows in the table and see if you understand where the number comes from.

Cups n	Revenue $r(n)$	Cost $c(n)$	Profit $p(n)$	
10	\$ 90	\$ 190	−\$ 100	Loss
20	\$ 180	\$ 230	−\$ 50	Loss
30	\$ 270	\$ 270	\$ 0	Break-Even
40	\$ 360	\$ 310	+\$ 50	Profit
50	\$ 450	\$ 350	+\$ 100	Profit
60	\$ 540	\$ 390	+\$ 150	Profit

Writing down the revenue, cost, and profit functions (but not writing a table, like in the previous box) is the start of a process called "Break-Even Analysis," which we will study in detail on Page 360.



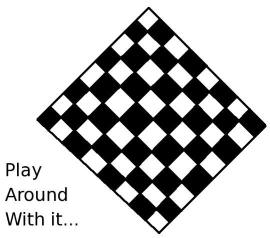
1-6-33

Let us suppose that your friend Beatrice has survived *Organic Chemistry I & II* and wishes to make money by selling a flashcard set that she's designed. (People in those classes have to memorize lots of complicated molecules and their chemical names.) You're going to help her make her business plan. Each set costs \$ 10 to print at the copy center, after a one-time-only design and layout fee of \$ 150. The online auction website she's using charges a \$ 3 fee plus 2% of the sales price for each set sold. Each set can be mailed for \$ 2.95 via Flat Rate Priority Mail, and she's selling them for \$ 24.95 each. We must now compute the cost, revenue, and profit functions for Beatrice's business.

I will get you started. The cost function would be

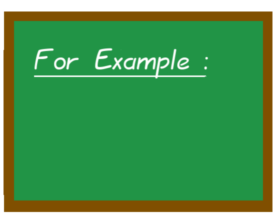
$$c(n) = \underbrace{150}_{\text{design}} + \underbrace{10n}_{\text{copies}} + \underbrace{(3 + (0.02)(24.95))n}_{\text{auction}} + \underbrace{2.95n}_{\text{shipping}}$$

You will figure out the rest in the next box.



1-6-34

- How would you simplify the cost function? [Answer: $c(n) = 150 + 16.449n$.]
- What is the revenue function? [Answer: $r(n) = 24.95n$.]
- What is the profit function? [Answer: $p(n) = 8.501n - 150$.]



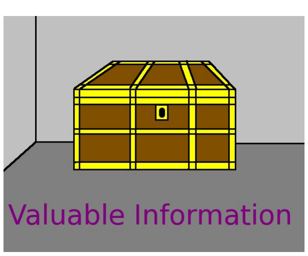
1-6-35

One practical use of subtracting functions is to show how much money someone would save by switching from one option or product to another. Let's consider the color printers from the real estate brochure example on Page 175. We'll compare the two most extreme printers, Printer A and Printer D. We had the functions

$$a(n) = 59 + 1.78n \text{ and } d(n) = 3999 + 0.38n$$

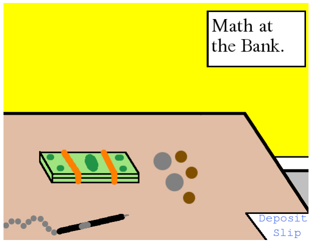
Therefore, the difference of the two functions (let's call it $s(n)$ for savings) would be

$$\begin{aligned} s(n) &= a(n) - d(n) \\ &= (59 + 1.78n) - (3999 + 0.38n) \\ &= 59 + 1.78n - 3999 - 0.38n \\ &= -3940 + 1.4n \end{aligned}$$



In problems similar to the previous box, students often do not know which function to put on which side of the minus sign. You should always put the *switching from* function first, and the *switching to* function second.

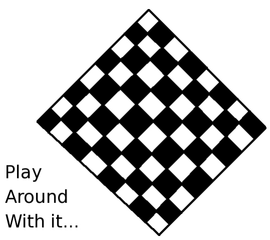
This is easy to remember, because it is chronological order.



We again summarize with a table:

How Many n	Printer A $a(n)$	Printer D $d(n)$	Savings $s(n)$	
1000	\$ 1839	\$ 4379	−\$ 2540	Loss!
1500	\$ 2729	\$ 4569	−\$ 1840	Loss!
2000	\$ 3619	\$ 4759	−\$ 1140	Loss!
2500	\$ 4509	\$ 4949	−\$ 440	Loss!
3000	\$ 5399	\$ 5139	+\$ 260	Profit.
3500	\$ 6289	\$ 5329	+\$ 960	Profit.
4000	\$ 7179	\$ 5519	+\$ 1660	Profit.

You can just pick a few rows in the table and see if you understand where the number comes from. Note that this table reinforces our conclusions from before, that Printer D is inherently very expensive and would only be right for businesses that anticipate producing a very large number of brochures. Nonetheless, if 4000 brochures were printed over the life of the printer, that \$ 1660 savings is very substantial.



1-6-36

Now let's examine the case of the calling card versus the land-line, from Page 176. We had the functions

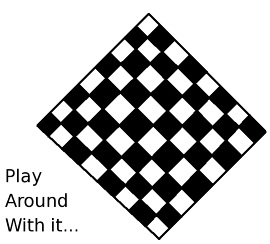
$$\ell(t) = 0.10 + 0.15t \text{ and } c(t) = 0.75 + 0.03t$$

Use them to answer the following questions:

- Write a function $s(t)$ to represent how much you would save by using the calling card instead of using the land-line. [Answer: $s(t) = -0.65 + 0.12t$.]
- What is $s(60)$? [Answer: 6.55, or \$ 6.55.]

Note: this should make sense, because earlier we calculated that using the land-line would cost \$ 9.10 while using the card would cost \$ 2.55, making a savings of

$$9.10 - 2.55 = 6.55$$



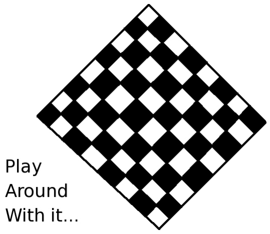
1-6-37

Let's check our work from the previous box.

- What is $\ell(10)$? [Answer: 1.6, or \$ 1.60.]
- What is $c(10)$? [Answer: 1.05, or \$ 1.05.]
- What is $s(10)$? [Answer: +0.55, or \$ 0.55.]

As a final check, observe that $\ell(10) - c(10) = 1.6 - 1.05 = 0.55 = s(10)$, as anticipated.

Just a reminder, that because this person would be switching *from* a land-line *to* a calling card, we should do $\ell(t) - c(t)$ and not $c(t) - \ell(t)$.



1-6-38

This one is a bit of a toughie. According to The World Factbook, available on the web through the website of the US Central Intelligence Agency (the CIA), there are approximately 2,578,840 births per week worldwide and approximately 1,077,930 deaths per week, as of July 1st, 2013. For short-term estimates, it is reasonable to assume that these rates will remain constant.

Consider the following:

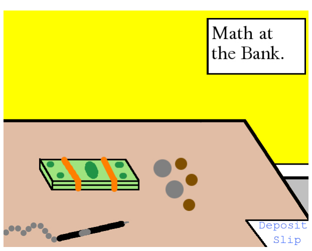
- Let w signify the number of weeks since July 1st, 2013.
- That means $w = 1$ is one week after July 1st, 2013, namely July 8th, 2013.
- That means $w = 2$ is two weeks after July 1st, 2013, namely July 15th, 2013.
- That means $w = 3$ is three weeks after July 1st, 2013, namely July 22nd, 2013.
- The world population on July 1st, 2013 was estimated to be 7,095,217,980 people.

Now answer the questions in the following box.

Here are some questions about the data in the previous box.

- What function $b(w)$ represents the total number of births after July 1st, 2013, up to and including week w?
- What function $d(w)$ represents the total number of deaths after July 1st, 2013, up to and including week w?
- What function $p(w)$ represents the world population in week w?
- Using these models, approximately when will the world's population be 8 billion people? (That is, how many weeks after July 1st, 2013?)

The answers can be found on Page 193.



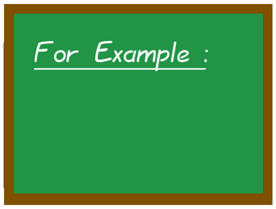
Now we're going to explore how to multiply a function by a constant. This comes up a lot in problems with units, which is especially important for international commerce, because different nations use different units for sizes, lengths, weights, and currency.

Suppose an Alaskan fishing conglomerate has hired you to analyze the potential takeover of an Icelandic fishing firm. You hire a careful statistician (as a consultant) from the fishing firm's home port, and he has computed a regression for the output of the largest fishing vessel in their fleet:

$$f_{kg}(t) = 6581 - 71t \text{ kg/year}$$

where t is the year, with $t = 0$ representing the year 2000.

The problem is that this function is giving an answer in terms of kilograms per year. Your boss's boss is unfamiliar with the metric system and has no idea what this means. In the next box, we will convert this function into one more comprehensible to someone who is not familiar with the metric system.



1-6-39

Continuing with the previous box, we have the function

$$f_{kg}(t) = 6581 - 71t \text{ kg/year}$$

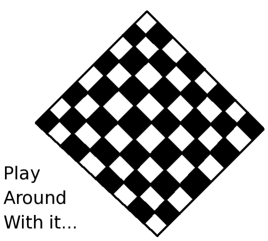
in metric units, which we'd like to convert to US Customary units. A quick internet search reveals that one kilogram is $2.20462\cdots$ pounds.

With that in mind, we must multiply $f_{kg}(t)$ by $2.20462\cdots$ to put the function in terms of pounds.

$$\begin{aligned} f_{lb}(t) &= (2.20462\cdots)f_{kg}(t) \\ &= (2.20462\cdots)(6581 - 71t) \\ &= (2.20462\cdots)(6581) - (2.20462\cdots)(71t) \\ &= (14,508.6\cdots) - (156.528\cdots)t \end{aligned}$$

Let's check the calculation from the previous box.

- For the year 2011, just as an example, the old function says $f_{kg}(11) = 5800$ kilograms.
- Meanwhile, for $t = 11$ the new function says $f_{lb}(11) = 12,786.7\cdots$ pounds.
- Just to check, $5800 \times 2.20462 = 12,786.7\cdots$, which is the value we expect.

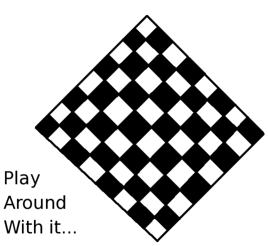


1-6-40

Similar to the analysis of the last three boxes, consider another vessel, slightly smaller, for which the model is

$$g_{kg}(t) = 5571 - 59t \text{ kg/year}$$

- Convert the model to use pounds. [Answer: $g_{lb}(t) = 12,281.9\cdots - (130.072\cdots)t$.]
- What does the old function say for the year 2011? [Answer: $g_{kg}(11) = 4922$ kg.]
- What does the new function say for the year 2011?
[Answer: $g_{lb}(11) = 10,851.1\cdots$ lbs.]
- Just to check, what is 4922×2.20462? [Answer: $10,851.1\cdots$ (excellent!)]



1-6-41

On Page 167 we saw the cost of printing n guidebooks represented by the function

$$c(n) = -\frac{n^2}{10,000} + 0.60n + 800 \text{ dollars}$$

which for convenience we can write as

$$c(n) = -0.0001n^2 + 0.60n + 800 \text{ dollars}$$

and note while this function appears mysterious, even arbitrary, we will see its derivation on Page 1212.

What if we were asked to write $c(n)$ in terms of Yen? A quick internet search reveals that on September 1st, 2013, one dollar was $99.4741\cdots$ Yen. What would our new function look like?

[Answer: $c(n) = (-0.00994741\cdots)n^2 + (59.6844\cdots)n + 79,579.2\cdots$ Yen.]

Play Around With it...

1-6-42

Let's consider again the $c(n)$ function of the previous box. What would this function be if we were asked to write it in terms of Euros? A quick internet search reveals that on September 1st, 2013, one Euro was $1.31903\cdots$ dollars.

[Answer: $c(n) = (-0.0000758132\cdots)n^2 + (0.454879\cdots)n + 606.506\cdots$ Euros.]

Applications to Science !

The famous magazine *Consumer Reports* publishes detailed reports, each year, on every make and model of cars. This is a great tool for people trying to choose a car. They also list the car's stopping distance at 60 mph. This is the distance that the vehicle will travel from the time you slam on the breaks, starting at 60 mph, until a speed of 0 mph is reached.

What you might not know is that county sheriffs and state troopers can use this data to determine how fast a car was going just prior to an accident, by measuring the black tire marks made upon the pavement when the car skidded to a halt.

The formula is

$$s = (60)\sqrt{\frac{D_x}{D_{60}}}$$

where D_{60} is the stopping distance at 60 mph, and D_x is the length of the tire marks on the pavement as measured. Both distances can be measured in any unit that you want, as long as they are both in the same units and s is measured in mph.

Play Around With it...

1-6-43

Continuing with the previous box, let's now imagine that at a law enforcement conference, an American sheriff makes friends with some counterparts from Latvia. He wants to share this very handy formula, but Latvians, using the metric system, measure land speed in kilometers per hour, not miles per hour. Can you convert the formula from the previous box into one that uses kilometers per hour?

A quick internet search reveals that 1 mile is 1.609 km, and so 1 mph is 1.609 kilometers per hour.

$$\left[\text{Answer: } s = (96.54)\sqrt{\frac{D_x}{D_{60}}}\text{km/hour.}\right]$$

Play Around With it...

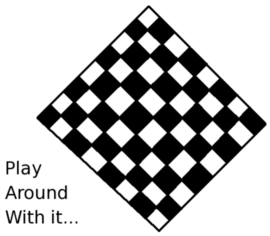
1-6-44

Continuing with the previous two boxes, the foreign friends from Latvia have emailed the American sheriff asking him to consult on the following highway accident:

The car is a Volvo C30, model year 2010, and the *Consumer Reports* website says the 60-mph-stopping distance is 143 feet. The tire marks on the pavement measure 116 meters. A quick internet search reveals that this is $380.577\cdots$ feet.

- Using the original formula, what was the Volvo's speed before it slammed on the brakes? [Answer: $97.8823\cdots$ mph.]
- Using the fact that 1 mph is 1.609 km/hour, convert the answer from the previous bullet into units that the Latvians will understand. [Answer: $157.492\cdots$ km/hour.]
- Using the new formula, what does it predict for the speed of the Volvo in km/hour? [Answer: $157.492\cdots$ km/hour.]

It's always nice when things come out exact!



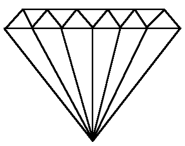
1-6-45

This problem is a very serious challenge, so you must not be dismayed if you can't get it just yet. We'll construct a model for a company that sells paper by the ream. We'll see this exact situation again on Page 388 of the module "Cost Benefit Analysis."

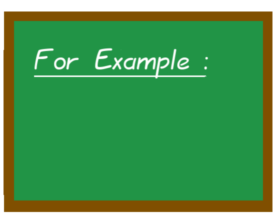
Company A sells stationary at the following rate: $ 1000 for the first 100 reams, and then $ 8 for each additional ream. You know that you'll always be ordering more than 100 reams. Write a function $a(n)$ that is the cost of ordering n reams from Company A.

The answer will be found on Page 193 of this module.

Hard but Valuable!



The following example is too hard. I realized this after I wrote it, however. Nonetheless, I thought I might leave it in the textbook. The reason is that if you're going to be doing business internationally (and these days, everyone is), then you're just going to have to deal with the confusion of different units of measure in the workplace. This problem will illustrate how serious that confusion can get. Only the bravest students should read past this point; for everyone else, the module is now concluded.



1-6-46

A particular industrial water heater says in its manual that it will boil water at the rate

$$r(c) = \frac{859.845}{100 - c} \text{ kilograms/hour}$$

where c is the temperature of water going into the machine (measured in °C). However, it is hard to think about large quantities of water in terms of kilograms, and most of the workers aren't expert in the metric system. Therefore, it would be great to convert this function to gallons per hour. A quick internet search reveals that 1 gallon of water is 3.78541 kilograms.

The main trick here is to figure out if we should multiply by 3.78541 or if we should divide. A gallon is several kilograms. However many kilograms this device can boil, surely the number in gallons will be smaller. Therefore, we divide. (An analogy: a week is several days; however many days are required to do a task, the number of weeks required is smaller, thus you divide by 7.)

To divide $r(c)$ by 3.78541, we will divide "each" coefficient by 3.78541. Here, there is only one coefficient, namely 859.845. Thus we divide 859.845 by 3.78541 and get 227.147. Our final answer is

$$r(c) = \frac{227.147}{100 - c} \text{ gallons/hour}$$

but this function is still imperfect. We still have the problem that c is temperature expressed in °C, whereas most factory workers are familiar with °F.

Now take a moment to think how you might convert the above function (from the previous box) into a function that uses degrees Fahrenheit. This is hard, so do not be discouraged at all if you do not get it. The solution is in the next box.

The key to the conversion I suggested in the previous box is to remember the relationship between Fahrenheit and Celsius, which is given by $f = (9/5)c + 32$. First, we should solve this for c. We get that

$$\begin{aligned} f &= \frac{9}{5}c + 32 \\ f - 32 &= \frac{9}{5}c \\ \frac{5}{9}(f-32) &= \frac{5}{9}\cdot\frac{9}{5}c \\ (5/9)(f-32) &= c \end{aligned}$$

We will continue in the next box.

Next, we plug that into the original and get

$$\begin{aligned} r(c) &= \frac{859.845}{100-c} \\ &= \frac{227.147}{100-(5/9)(f-32)} \\ &= \frac{227.147}{100-[(5/9)f-(5/9)(32)]} \\ &= \frac{227.147}{100-(5/9)f+(5/9)(32)} \end{aligned}$$

Note: Having the (5/9) there is very inconvenient, so let's multiply numerator and denominator by (9/5), which will turn the (5/9) into 1.

$$\begin{aligned} &= \frac{227.147}{100-(5/9)f+(32)(5/9)}\cdot\frac{(9/5)}{(9/5)} \\ &= \frac{227.147(9/5)}{100(9/5)-(9/5)(5/9)f+32(5/9)(9/5)} \\ &= \frac{408.864}{180-f+32} \\ &= \frac{408.864}{212-f} \end{aligned}$$

Our final answer is

$$r(f) = \frac{408.864}{212-f} \text{ gallons/hour}$$

By the way, does the 212 look familiar? It really should look very familiar to you. The boiling point of water is 212°F.

After a calculation that massive, we really need to check our work to make sure that nothing has gone horribly wrong. We start with the observation that 68°F is equal to 20°C. Then we ask the original function how quickly water will be boiled at 20°C, and the new function how quickly water will be boiled at 68°F.

For the original, we have

$$c(20) = \frac{859.845}{100-20} = \frac{859.845}{80} = 10.7480\cdots \text{ kilograms/hour}$$

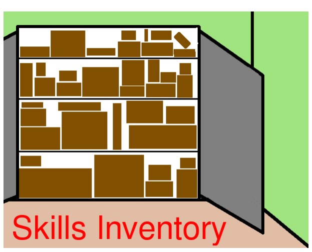
For the new function, we have

$$f(68) = \frac{408.864}{212-68} = \frac{408.864}{144} = 2.83933\cdots \text{ gallons/hour}$$

Last but not least, we convert the gallons per hour into kilograms per hour by recalling that 1 gallon of water is 3.78541 kilograms.

$$2.83933\frac{\text{gallons}}{\text{hour}} \cdot \frac{3.78541 \text{ kg}}{1 \text{ gallon}} = (2.83933)(3.78541)\frac{\text{kg}}{\text{hour}} = 10.7479\cdots \text{ kg/hr}$$

which is phenomenal accuracy.



We have reviewed the following skills in this module.

- how to evaluate a function;
- how to convert words in a word problem into a function;
- how to add two functions;
- how to combine two regressions to make a more general regression, by adding them;
- how to compute the relative error of a regression;
- how to negate a function;
- how to subtract a function from another function;
- how to compute simple cost, revenue, and profit functions;
- how to compute the savings from switching between products, by subtracting functions;
- how to multiply a function by a constant;
- how to perform international unit conversion of a function, by multiplying it by a constant;
- how to check our work in several of these cases by plugging in the inputs 0, 1, and 2;
- the vocabulary terms input and output;
- plus "early mention" of some vocabulary terms to be defined later: best-fit line, domain, economy of scale, extrapolation, line of best fit, regression, and trend line.

Here are the answers to the problem of forecasting the world's population growth, from Page 187.

- What function $b(w)$ represents the number of births since July 1st, 2013? [Answer: $b(w) = 2,578,840w$.]
- What function $d(w)$ represents the number of deaths since July 1st, 2013? [Answer: $d(w) = 1,077,930w$.]
- What function $p(w)$ represents the population w weeks after July 1st, 2013?

$$\begin{aligned} p(w) &= 7,095,217,980 + b(w) - d(w) \\ &= 7,095,217,980 + 2,578,840w - 1,077,930w \\ &= 7,095,217,980 + 1,500,910w \end{aligned}$$

The last answer is given in the next box.

Continuing from the previous box...

- Using these models, approximately when will the world's population be 8 billion people? (In other words, how many weeks after July 1st, 2013?)

$$\begin{aligned} p(w) &= 8,000,000,000 \\ 7,095,217,980 + 1,500,910w &= 8,000,000,000 \\ 1,500,910w &= 904,782,020 \\ w &= 904,782,020/1,500,910 \\ w &= 602.822\cdots \end{aligned}$$

- If you are curious, $602.822\cdots$ weeks is 11.5927 years, or approximately 11 years and 216 days.

Our calculation in the previous box would imply a date of February 2nd, 2025. However, using a linear function for population growth is unwise, since the world's population growth rate is better represented by an exponential function. (We'll learn about exponential functions on Page 474.) Linear functions underestimate exponential growth functions, so the 8-billion mark will almost surely come sooner than the year 2025. However, long-term predictions are always risky, because human behavior is often difficult to predict.

Here is the solution to the modeling of the company that sells paper by the ream. That question was on Page 190 of this module.

Company A sells stationary at the following rate: \$ 1000 for the first 100 reams, and then \$ 8 for each additional ream. You know that you'll always be ordering more than 100 reams.

For Company A, we know that we'll be using more than 100 reams, so we'll pay \$ 1000 for the first 100 reams, and order $n - 100$ additional reams at \$ 8 each. That means

$$a(n) = 8(n - 100) + 1000$$

which can be simplified to

$$a(n) = 8n + 200$$