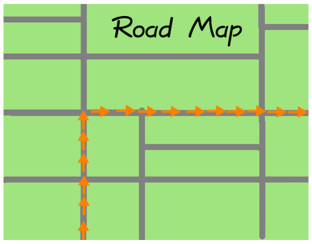
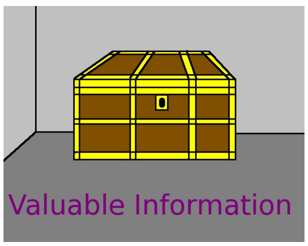
Module 1.8: Review of Basic Inequalities

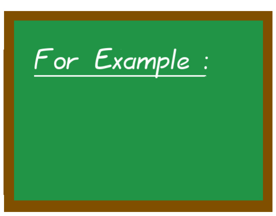


Chances are that you've worked with inequalities in a previous algebra class, either in high school or in the university setting. However, I've found that it is really productive to review inequalities from scratch. While inequalities are a minor topic in algebra coursework, they're an extremely major topic in economic, commercial, and industrial modeling. Accordingly, we'll be using them often in this book, and I want to make sure that everyone's memory is fully refreshed.



These are the basic rules for using inequalities:

- You can add anything to both sides.
- You can subtract anything from both sides.
- You can multiply any number n (other than zero) to both sides, but...
  - If n is positive, then you do not flip the inequality sign.
  - If n is negative, then you must flip the inequality sign.
- You can divide both sides by any number m (other than zero), but...
  - If m is positive, then you do not flip the inequality sign.
  - If m is negative, then you must flip the inequality sign.



1-8-1

Let's suppose we are challenged with solving

$$3x + 5 \leq 11$$

Our goal is to get x alone on one side of the inequality. First, we subtract 5 from each side to get

$$3x \leq 6$$

Next, we divide both sides by 3 to get

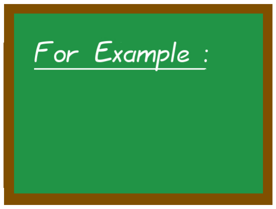
$$x \leq 2$$

Finally, we conclude that $3x + 5 \leq 11$ whenever $x \leq 2$.

Now let's check our work by plugging in a few numbers in for x. Consider 0, 1, 2, 3, 4, and 5. We obtain

if $x = 0$	then	$3x + 5 = 5$	if $x = 3$	then	$3x + 5 = 14$
if $x = 1$	then	$3x + 5 = 8$	if $x = 4$	then	$3x + 5 = 17$
if $x = 2$	then	$3x + 5 = 11$	if $x = 5$	then	$3x + 5 = 20$

Indeed, it seems to make sense that whenever $x \leq 2$, it is true that $3x + 5 \leq 11$.



1-8-2

Suppose we're asked to determine when

$$80 - 3x \geq 20$$

We could begin by subtracting 80 from each side to get

$$\begin{aligned} -3x &\geq 20 - 80 \\ -3x &\geq -60 \end{aligned}$$

Now we want to divide both sides by -3. However, we must also flip the direction of the inequality symbol when we do so (as dictated by the rules on the first page of this module)

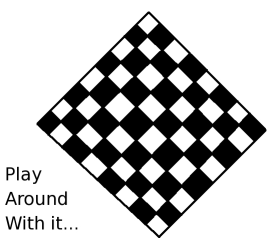
$$\begin{aligned} x &\leq (-60/-3) \\ x &\leq 20 \end{aligned}$$

Finally, we conclude that $80 - 3x \geq 20$ whenever $x \leq 20$.

As before, the easiest way to check our work above is to plug a few numbers in for x. Consider 0, 10, 20, 30, 40, and 50. We obtain

if $x = 0$	then	$80 - 3x = 80$	if $x = 30$	then	$80 - 3x = -10$
if $x = 10$	then	$80 - 3x = 50$	if $x = 40$	then	$80 - 3x = -40$
if $x = 20$	then	$80 - 3x = 20$	if $x = 50$	then	$80 - 3x = -70$

This appears to be consistent with our claim that if $x \leq 20$ then $80 - 3x \geq 20$. Likewise, if $x > 20$ then $80 - 3x < 20$.



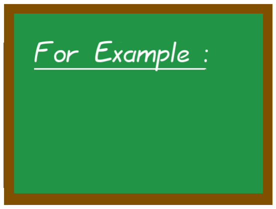
1-8-3

Just to make sure that we've got this skill down pat, let's try four more examples.

- When is $8x + 9 < 65$? [Answer: $x < 7$.]
- When is $12 - 8x > -4$? [Answer: $x < 2$.]
- When is $18 - 5x \leq 3$? [Answer: $x \geq 3$.]
- When is $95x + 15 \geq 2675$? [Answer: $x \geq 28$.]

I know it seems hard to believe that algebra this symbolic and abstract can be in any way whatsoever related to finance or economics. Nonetheless, in later sections we will have many practical and important uses of inequalities in industrial and commercial situations.

My personal favorite, and that of many students, is the Cost-Benefit Analysis module which starts on Page 386.

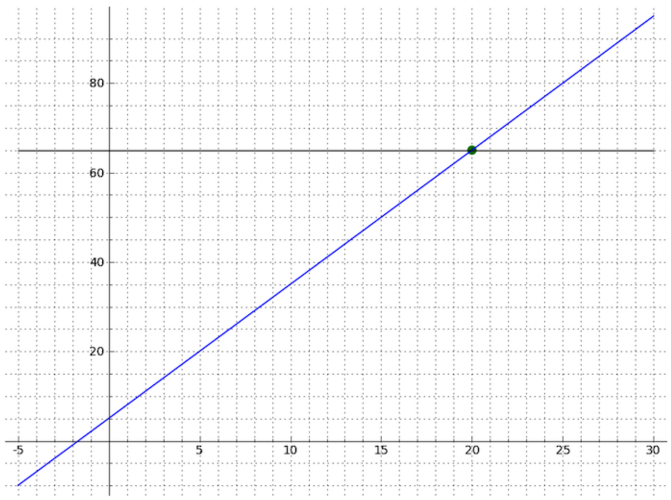


1-8-4

Let's suppose that $f(x) = 3x+5$, and that we must find when $f(x) \leq 65$. We would proceed as follows:

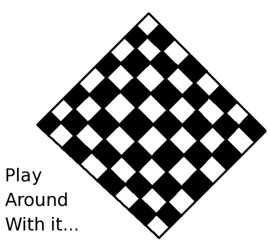
$$\begin{aligned} f(x) &\leq 65 \\ 3x+5 &\leq 65 \\ 3x &\leq 60 \\ x &\leq 20 \end{aligned}$$

Therefore, we learn that $f(x) \leq 65$ whenever $x \leq 20$. Likewise $f(x) > 65$ whenever $x > 20$.



This plot is a graphical view of the problem from the previous box. As you can see, there is a horizontal line at $y = 65$. The diagonal line is the function $f(x) = 3x + 5$, the function we were studying in the previous box. We've drawn a large dot where the two lines cross each other.

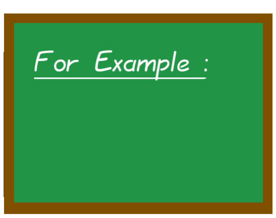
As you can see, the lines cross at $x = 20$. To the right of the crossing, the function is giving y-values bigger than 65. Meanwhile, to the left of the crossing, the function is giving y-values smaller than 65. Accordingly, we write that $f(x) \leq 65$ whenever $x \leq 20$. This also implies $f(x) > 65$ whenever $x > 20$.



1-8-5

Before we solve a business-related word problem using this technique, let's try two quick questions for practice.

- If $f(x) = 8 - 2x$ then when is $f(x) \leq 5$? [Answer: when $x \geq 1.5$.]
- If $f(x) = 19x + 15$ then when is $f(x) > 157.5$? [Answer: when $x > 7.5$.]



1-8-6

Suppose that you are working for a small recreational vehicle dealership. The dealership assembles ATVs from kits and sells them to interested consumers. A new type of tire equipped with some advanced safety feature has recently entered the market and is extremely popular. Your dealership has 1000 of this tire in stock, and each ATV requires 4 tires.

A big order for an ATV convention is currently being negotiated. While you have ample parts on hand to assemble many ATVs, there are only 1000 of the new tires in stock and only \$ 3400 of cash reserves available for the purchase of more tires, which cost \$ 17 each. What function will say how much money must be spent on new tires in order to build a ATVs? If the entire \$ 3400 cash reserve is spent on new tires, how many ATVs can be built?

The solution is given in the next box.

Now we will solve the problem from the previous box. First, we should write a function down. We know that a ATVs would require $4a$ tires. We have 1000 tires in stock, so we must order $4a-1000$. This means that we would spend

$$f(a) = 17(4a - 1000)$$

dollars on new tires. Recall that we have \$ 3400 in cash reserves—that is the maximum amount that we can spend on new tires, thus $f(a) \leq 3400$.

We can now solve this inequality:

$$\begin{aligned} f(a) &\leq 3400 \\ 17(4a-1000) &\leq 3400 \\ 4a-1000 &\leq 3400/17 \\ 4a-1000 &\leq 200 \\ 4a &\leq 1200 \\ a &\leq 300 \end{aligned}$$

Finally, we conclude that we can build up to 300 ATVs using the 1000 tires available and the \$ 3400 cash reserve.

Play Around With it...

1-8-7

Let's explore some more scenarios with the ATV manufacturer.

- How about if only \$ 2397 can be spent on new tires? What is the final inequality in that case? [Answer: $a \leq 285.25$.]
- How about if \$ 4400 were in the cash reserves, but the price for one tire were \$ 16 instead of \$ 17? What is the final inequality in that case? [Answer: $a \leq 318.75$.]

Play Around With it...

1-8-8

Suppose that Bob the accountant is planning a party, and he wants everyone to have exactly 4 cans of soda. (Perhaps Bob has obsessive-compulsive disorder.) He has three 6-packs in the fridge, and additional 6-packs cost \$ 2.94 each at the corner store. Also, Bob has only \$ 80 of spare money on hand to spend on supplying the soda. If he wishes to invite x friends (including himself), then ...

- How many cans of soda will be consumed? [Answer: $4x$.]
- Of those, how many new soda cans must be purchased? [Answer: $4x-18$.]
- What is the cost of a can of soda? [Answer: 49 cents.]
- How much will the soda to be purchased cost? [Answer: $(0.49)(4x-18)$.]

The problem continues in the next box.

Play Around With it...

1-8-9

Continuing with the previous box. . . ,

- What inequality upper bounds this cost function by the quantity of funds available?

$$[\text{Answer: } (0.49)(4x - 18) \leq 80 \qquad \text{or} \qquad 1.96x - 8.82 \leq 80.]$$

- Solve the above inequality for x. [Answer: $x \leq 45.3163\cdots$.]
- What is the maximum number of friends that can be invited, under these restrictions? [Answer: 45.]
- Note: I hope you didn't answer 45.3163 for the previous bullet. You must round your answer down to the nearest integer, since people come in integer numbers, not fractions. You should not cut your friends into pieces. There can be severe legal consequences if you do that.
- If Bob does invite 45 people (including himself) how many six-packs should Bob buy? [Answer: 27 six-packs.]

The next problem is a bit massive, but that's okay. There's no point in a future economist, financier, accountant, marketing director, manager, industrial engineer, realtor or other business leader studying mathematics at all, unless realistic real-world problems are going to be attempted. Naturally, real-world problems have lots of data and are a bit complicated. Do not be alarmed if the next problem takes you a few minutes.

Play Around With it...

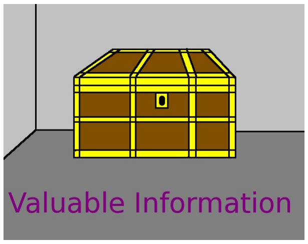
1-8-10

Imagine you are the marketing manager at a small company that makes desserts and snacks. An airline has suddenly decided to adopt your almond cookies product for sale on their flights. Naturally, you are delighted. They don't know yet how many they are going to order, so they're asking you how many you can produce, to be delivered in 60 days. However, upon consultation with your colleagues in logistics, you learn that the usual supplier of almond extract for your company has gone bankrupt—this is unfortunate timing. Luckily, your factory has 300 gallons on hand. Each package of almond cookies requires two fluid ounces of almond extract. Another supplier is willing to sell you almond extract for \$ 80 per gallon. The operating fund is a bit tight, having \$ 104,635 dollars available for the purchase of almond extract at this time, and there is not enough time to acquire a short-term loan at a reasonable rate of interest. Accordingly, you must now compute what range of orders your firm can fulfill in the short-term, and communicate this to the airline.

Note: a quick internet search reveals that there are 128 fluid ounces in one gallon. If the airline orders x packages of almond cookies, then . . .

- . . . how many fluid ounces of almond extract are to be consumed?
- . . . how many gallons of almond extract are to be consumed?
- . . . how many gallons of almond extract are to be purchased?
- . . . how much will that cost?
- . . . what inequality upper bounds that cost by the amount of money available?
- Solve this inequality for x.
- What is the largest order that can be fulfilled at this time?

Note: the answers will be given on Page 233.



In the previous box, we calculated that the number of packages of almond cookies ordered must be at most 102,908. In later modules we'll frequently write

$$0 \leq x \leq 102,908$$

to represent this situation, and to rule out negative values of x. Those symbols are meant to indicate that x is sandwiched between 0 and 102,908. It is useful to simply read it as "x is between 0 and 102,908" rather than try to decompose it into "0," "$\leq$," "x," "$\leq$," and "102,908" and mentally translate these symbols individually.

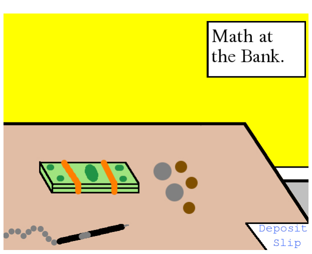
We include the $x = 0$ and $x = 102,908$ cases when it is written as above. If we had wished to exclude those cases, we would have written instead

$$0 < x < 102,908$$

You might have noticed that we can actually solve the almond-cookie problem without inequalities at all. This is done by slowly working backwards from a specific figure, and requires less sophisticated mathematics.

1. There are $104,635$ dollars available and almond extract costs \$ 80 per gallon, so at most $104,635 \div 80 = 1307.93\cdots$ gallons can be purchased.
2. In addition to the extract to be purchased, we have 300 gallons on hand, for a total of $1307.93\cdots + 300 = 1607.93\cdots$ gallons.
3. We have available $(1607.93)(128) = 205,816$ fluid ounces of almond extract.
4. The total number of almond-cookie packages we can produce is therefore $205,816 \div 2 = 102,908$.

Note: while this trick of "working backwards" was possible in this specific problem, we will see many problems throughout the textbook where it is not at all possible to solve the problem unless one works with inequalities.

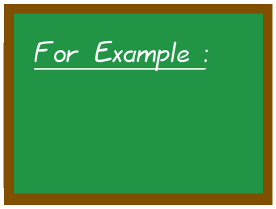


In the ATV problem from Page 223, the sole limiting factor was the number of tires available (both those in stock and those to be purchased). In the almond-cookie problem of the last four boxes, the sole limiting factor was the quantity of almond extract available. Is it realistic to imagine an industrial situation being governed by only one commodity?

Actually, yes—these things do occasionally happen, as we saw in those problems, that described reasonable business situations. However, it is much more often the case that three, four, or even more constraints are simultaneously limiting factors. This leads to an amazingly powerful subject to which I devote an entire chapter of this textbook, "Inequalities and Industrial Engineering."

Solving problems where one is trying to maximize profit, minimize cost, or maximize revenue, while staying within all given constraints is called *linear programming*, among mathematicians. On the other hand, in government it is called *operations research* and in the business world it is called *industrial engineering*.

Now we are finished examining the short-term prospects of the almond cookie manufacturer. This problem also comes up on Page 1280, where we examine the long-term influence of the price of almond extract on each package of cookies. At this time, we need to step back to symbolic mathematics for only two boxes, before we return to modeling interesting commercial and industrial situations.

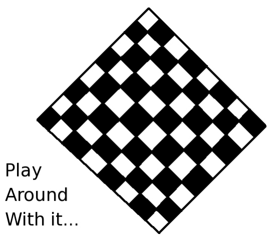


1-8-11

Let $f(x) = 3x + 7$ and $g(x) = 10 - 2x$. Let's suppose that we were interested in finding out when $f(x)$ is less than or equal to $g(x)$. Here's how we could compute that:

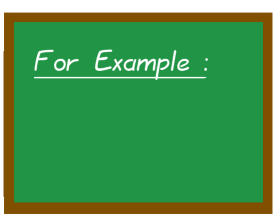
$$\begin{aligned} f(x) &\leq g(x) \\ 3x + 7 &\leq 10 - 2x \\ 3x &\leq 3 - 2x \\ 5x &\leq 3 \\ x &\leq 3/5 \end{aligned}$$

Therefore, we know that if $x \leq 3/5$, then $f(x) \leq g(x)$. Accordingly, if x is not less than or equal to $3/5$, which means $x > 3/5$, then it will not be the case that $f(x) \leq g(x)$—which means $f(x) > g(x)$.



1-8-12

- If $a(x) = 12x - 600$ and $b(x) = 3x + 900$, then for what values of x is $a(x) > b(x)$? [Answer: $x > 166.6\overline{6}$, which can also be written as $166.6\overline{6} < x$.]
- If $g(x) = 200 - 5x$ and $h(x) = 100 - 2x$, then when is $g(x) < h(x)$? [Answer: $x > 33.3\overline{3}$, which can also be written as $33.3\overline{3} < x$.]



1-8-13

There is a problem that we're going to examine later in the book (on Page 307), which models how the subscribers of a magazine migrate from the print version to the web-based version over time. This problem covers the years $t = 1998$ until $t = 2009$. When we reach that problem in the book, you will calculate that the print subscribers are modeled by

$$p(t) = -49.5t + 99,451$$

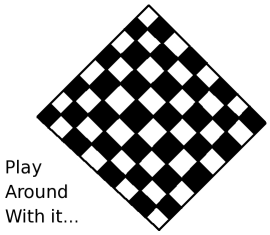
while the web-based subscribers are modeled by

$$w(t) = 61t - 121,878$$

According to the model, when will the number of web-based subscribers have exceeded the number of print-based subscribers?

$$\begin{aligned} w(t) &> p(t) \\ 61t - 121,878 &> -49.5t + 99,451 \\ 61t &> -49.5t + 221,329 \\ 110.5t &> 221,329 \\ t &> 2002.97\cdots \end{aligned}$$

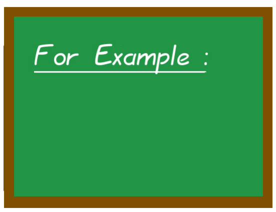
Therefore, the answer is that after very late in 2002 (essentially 2003 and afterward), the number of web-based subscribers would have exceeded the number of print-based subscribers, according to the model.



1-8-14

Consider some variations on the previous box:

- Perhaps you want to find out when the model predicts that the number of print subscribers would be double or more than the number of web-based subscribers. What inequality would you solve? [Answer: $p(t) > 2w(t)$.]
- Solve that inequality for t. [Answer: $t < 2001.20\cdots$.]
- Likewise, to find out when the model predicts that the number of web-based subscribers will have exceeded triple the number of print-based subscribers, what inequality would you solve? [Answer: $w(t) > 3p(t)$.]
- Solve that inequality for t. [Answer: $t > 2005.87\cdots$.]



1-8-15

Suppose two high school students are studying for a math competition, and they have the following question: "When is the area of a circle greater than or equal to its perimeter?" First, they will double check (with a quick internet search) that

$$A = \pi r^2 \qquad \text{and} \qquad C = 2\pi r$$

are the formulas for the area and circumference of a circle. These are actually functions of the radius, namely $A(r) = \pi r^2$ and $C(r) = 2\pi r$.

As you can see, this math competition question really is about comparing two functions. What Bob needs to find out is the values of r that will make $A(r) \geq C(r)$.

We explore the solution in the next few boxes.

The purpose of this example in the previous box is to make you aware of a serious pitfall. When working with inequalities, you must never divide by a variable (unless you know that it is always positive). To illustrate the danger of dividing by variables which might be negative or zero, we're going to see what happens if you do that.

$$\begin{aligned} A(r) &\geq C(r) \\ \pi r^2 &\geq 2\pi r \\ r^2 &\geq 2r \\ r &\geq 2 \qquad \leftarrow \text{DANGER: THIS STEP IS ILLEGAL} \end{aligned}$$

Now, let's see why this is wrong.

According to the (incorrect) calculation in the previous box, when $r \geq 2$, we expect $A(r) \geq C(r)$. Also, when $r < 2$ then we expect $A(r) < C(r)$. Sadly, these conclusions wrong.

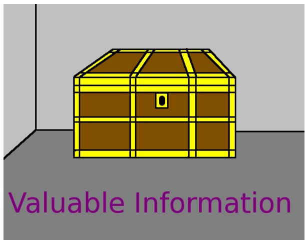
Consider the case of a simple point, which is a circle of radius zero. What is the area of a point? Of course it is zero, as we can see:

$$A(0) = \pi 0^2 = 0$$

Similarly, the circumference of a simple point is also zero. The following calculation points this out quite readily:

$$C(0) = 2\pi 0 = 0$$

Yet, according to our (incorrect) calculation, since $0 < 2$ then we expect $A(0) < C(0)$, as we wrote in the first paragraph of this very box. However, it is actually the case that $A(0) = C(0)$. Therefore, we know that something, somewhere is broken.

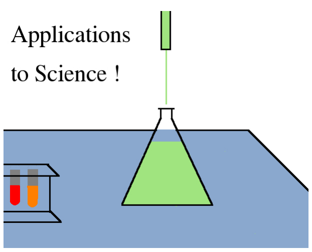


The moral of the story is that, when working with inequalities, you should never divide by a variable unless you know it is always positive.

Now that we know that we must never divide by a (potentially non-positive) variable, you might be curious how to actually solve the "area-circumference" problem. We'll learn that later, in the module "Polynomials and Inequalities," which begins on Page 803.

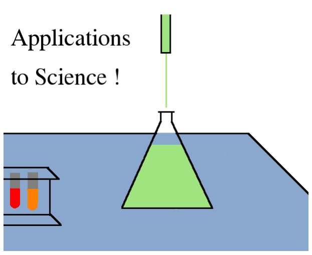
In all my years of grading mathematics quizzes and examinations, I've found that there are really only four categories of mistakes that comprise the vast majority of cases where students lose points. If you only remember these four valuable guidelines, then I think you'll score very well when working with inequalities.

- You must flip the sign if you multiply or divide by a negative number, but not if you multiply or divide by a positive number.
- You must never flip the sign while adding or subtracting.
- You must never divide by a variable (unless you know it is always positive).
- Be very careful about the units. (We'll explore this point in the next four boxes.)



Imagine that Alice had studied abroad in Spain, and had made friends with Fernando, who lives in Madrid. Years later, Alice wishes to explain to Fernando the problem about the area and circumference of a circle, and so in her next letter to him, she includes a pair of construction-paper circle cutouts. They have radii equal to 1 inch and 3 inches. As you can see, the smaller circle has an area of π and a circumference of 2π. On the other hand, the larger circle has an area of 9π and a circumference of 6π. Alice feels confident now because she has produced one circle where the area exceeds the circumference, and one circle where the circumference exceeds the area.

We will continue this story in the next box.

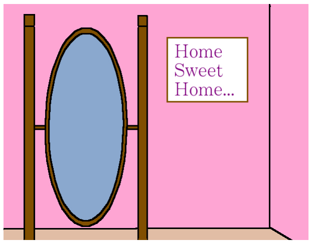


The story from the previous box now continues. Fernando is extremely confused upon receipt of the circles. He claims that both have an area that exceeds the circumference. The smaller circle has a radius of 2.54 cm, and that means its area is $(\pi)(2.54)^2 = 20.2682\cdots$ square centimeters, while the circumference is $(2)(\pi)(2.54) = 15.9592\cdots$ cm. As you can see, the area is larger than the circumference. Fernando then measures that the larger circle has a radius of 7.62 cm, and that means the area is $(\pi)(7.62)^2 = 182.414\cdots$ square centimeters and its circumference is $(2)(\pi)(7.62) = 47.8778\cdots$ centimeters. Yet again, the area exceeds the circumference.

Fernando decides to explain his confusion to Alice in a letter, and sends her two construction-paper cutouts of circles, of radius 1 cm and 3 cm. He then explains that, unlike the pair that she had sent him, these new circles indeed have one with the area larger and one with the circumference larger.

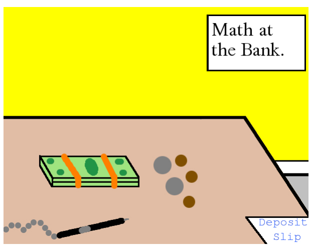
Now it is Alice's turn to be confused. To make a long story short, the smaller circle has a radius of $0.393700\cdots$ inches, an area of $0.486947\cdots$ square inches, and a circumference of $2.473695\cdots$ inches. On the other hand, the larger circle has a radius of $1.18110\cdots$ inches, an area of $4.38253\cdots$ square inches, and a circumference of $7.42108\cdots$ inches. Alice writes back to Fernando, telling him that he sent her two circles, both of which had a circumference larger than the area.

Who is right and who is wrong?

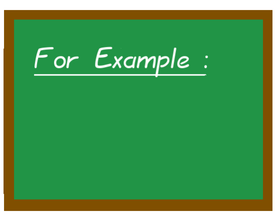


A Pause for Reflection...
What is the point of the previous box? Of course, part of it is to explain that if you wish to have trade, commerce or industry across cultures, then you have to be very careful about the units. However, there is also a more specific point.

Whenever comparing two quantities, it is imperative to ensure that the units are the same. If the units are not the same, then the comparison is meaningless. The possible units for circumference are centimeters, inches, meters, feet, and so on. On the other hand, the possible units for area are *square* centimeters, *square* meters, *square* feet, and so on. As you can see, it can never come to pass that you measure the circumference and area of a circle using the same units for each. Therefore, outside of pure mathematics, you should never compare an area with a circumference.



In my own experience, in my career before becoming a professor, I worked as an engineer for several years. When I was working for the Naval Surface Warfare Center, I had a relatively senior manager ask me what was larger: 300 pounds or 136 joules. I tried to explain that he was asking me to compare apples and oranges (or in this case, forces and energies), but he became very upset that I was not giving him a simple answer. This unpleasant event illustrates why every manager should have some small exposure to scientific matters.



1-8-16

You might find the area/circumference example to be a bit artificial. After all, the only place where the method went wrong was when $r = 0$, which is just one solitary point on the infinite number line. Accordingly, here's another example: "If $f(x) = x^2$ and $g(x) = 3x$, then when is $f(x) > g(x)$?"

$$\begin{aligned} f(x) &> g(x) \\ x^2 &> 3x \\ x &> 3 \qquad \leftarrow \text{DANGER: THIS STEP IS ILLEGAL} \end{aligned}$$

Based off this (incorrect) calculation, we'd expect that if $x > 3$ then $f(x) > g(x)$, and likewise if x is not greater than 3 then $f(x)$ is not greater than $g(x)$.

A quick way to determine how the previous calculation has gone wrong is to simply plug some values of x into both functions and see which is bigger.

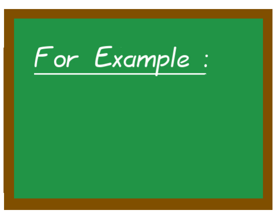
x	$f(x)$	$g(x)$	$f(x) > g(x)$?	x	$f(x)$	$g(x)$	$f(x) > g(x)$?
-4	16	-12	yes	2	4	6	no
-3	9	-9	yes	3	9	9	no
-2	4	-6	yes	4	16	12	yes
-1	1	-3	yes	5	25	15	yes
0	0	0	no	6	36	18	yes
1	1	3	no	7	49	21	yes

As you can see, the answer $x > 3$ is very wrong because of $x = -4$, $x = -3$, $x = -2$, and also $x = -1$.

In the example about the area and circumference of a circle, we saw that the variable being zero caused a problem. In the example about $f(x) = x^2$ and $g(x) = 3x$, we saw that the variable being negative caused a problem.

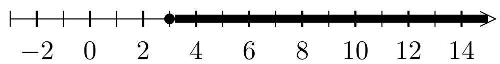
Once again, we are reminded: when working with inequalities, you should never divide by a variable unless you know it is always positive.

Sometimes, but not very often, we wish to graph the solution to an inequality on the number line. This can help clarify what is meant, as the human eye can interpret a graph more easily than it can interpret a formula. In the next two boxes we'll see some examples.

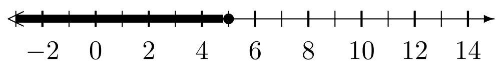


1-8-17

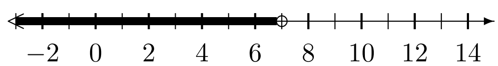
This is how you would draw $x \geq 3$:



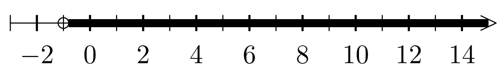
This is how you would draw $x \leq 5$:

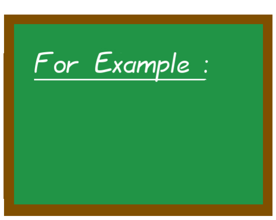


This is how you would draw $x < 7$. (Notice how the filled circle from above is now replaced by an unfilled circle, indicating that x cannot equal seven.)



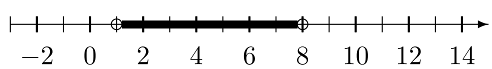
This is how you would draw $x > -1$. (Again, we see that the filled circle from above is now replaced by an unfilled circle, since x cannot equal -1.)



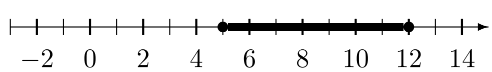


1-8-18

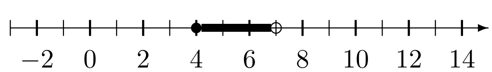
This is how you would draw $1 < x < 8$:

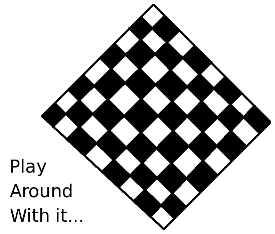


This is how you would draw $5 \leq x \leq 12$:



You can even mix the two types of inequalities. I've never seen this actually come up, but if you wanted to plot $4 \leq x < 7$ for some strange reason, then it would look like this:





1-8-19

Try to draw some number-line sketches for the following inequalities:

- Plot $x > 5$.
- Plot $2 \leq x \leq 4$.
- Plot $3 < x < 6$.
- Plot $-1 < x \leq 7$.

The answers will be found on Page 233.

The previous three boxes talk about a way of plotting inequalities that perhaps does not sound very important. However, this notation can be useful, and we'll use it later in this book.

At this point, I'd like to show you one of my favorite examples that shows how complicated inequalities can get. Suppose someone wants to analyze the following polynomial:

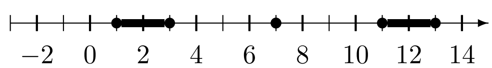
$$a(x) = x^6 - 42x^5 + 683x^4 - 5404x^3 + 21,303x^2 - 37,562x + 23,021$$

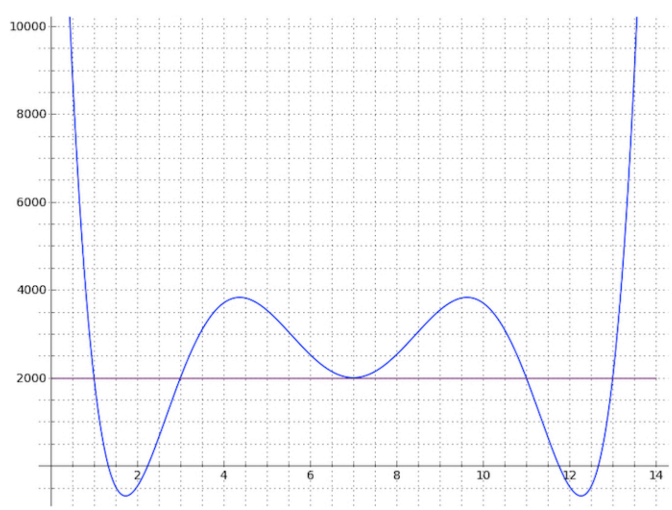
and furthermore, they are interested in learning when $a(x) \leq 2000$.

As it turns out, the answer is

$$1 \leq x \leq 3 \quad \text{or} \quad x = 7 \quad \text{or} \quad 11 \leq x \leq 13$$

which we would draw as



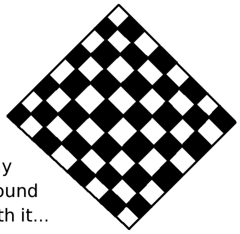


In this diagram you can see what is really going on. The blue curve is the polynomial $a(x)$ that we are talking about in the previous box. The purple horizontal line represents $y = 2000$. Using the grid, it is pretty clear that $a(x) \leq 2000$ whenever $1 \leq x \leq 3$ as well as $11 \leq x \leq 13$. Also, if you look carefully, you can see that $a(x) = 2000$ when $x = 7$ exactly.

This is not the kind of analysis that we are concerned with in this book. I mention it only so that you know that not all problems involving inequalities are easy. In fact, some problems involving inequalities are quite challenging and complicated.

A mathematics major would study these topics in detail in a course often called *Real Analysis*.

Our review of inequalities is over for now. We will have frequent use of these important skills in the coming chapters. Let's now close our exploration with an interesting commercial problem, and then will follow a pair of boxes with the answers to some previous checkerboards.



Play
Around
With it...

1-8-20

Suppose you are working with a warehouse, and learn that shipping some important sprocket will cost \$ 8.96 per sprocket, plus a flat \$ 39.96 fee. Due to a special offer, you have 25% off shipping costs for today only. In order to make maximum use of this offer, you wish to ship a very large number of sprockets. However, you have only \$ 896.72 in the cash reserves at this time.

- What function $s(n)$ gives the normal cost for shipping n sprockets? [Answer: $s(n) = 8.96n + 39.96$.]
- What function $t(n)$ gives the cost today, with the 25% off offer? [Answer: $t(n) = (0.75)(8.96n + 39.96)$, or equivalently $t(n) = 6.72n + 29.97$.]
- What inequality represents spending at most \$ 896.72? [Answer: $t(n) \leq 896.72$.]
- What values of n will satisfy this inequality? [Answer: $n \leq 128.980\cdots$.]
- How many sprockets can be ordered? [Answer: up to 128.]

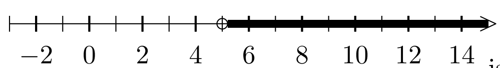
Here are the answers to the almond-cookie problem from Page 225. If the airline orders x packages of almond cookies, then ...

- ...how many fluid ounces of almond extract are to be consumed? [Answer: $2x$.]
- ...how many gallons of almond extract are to be consumed? [Answer: $x/64$.]
- ...how many gallons of almond extract are to be purchased? [Answer: $x/64 - 300$.]
- ...how much will that cost?

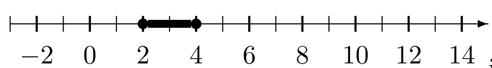
  [Answer: $(80)(x/64 - 300)$ or $(80/64)x - 24,000$ or $(5/4)x - 24,000$.]

- ...what inequality upper bounds that cost by the amount of money available? [Answer: $(5/4)x - 24,000 \leq 104,635$.]
- We obtain $x \leq 102,908$, which can also be written as $102,908 \geq x$.
- Thus a maximum order of 102,908 packages can be fulfilled at this time.

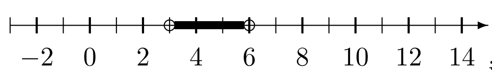
Here are the number-line sketches which you were asked to draw on Page 232:



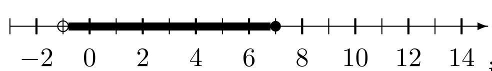
is the plot for $x > 5$



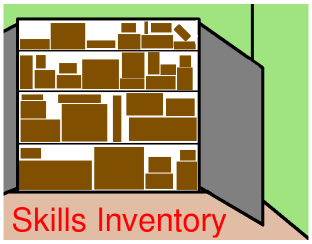
is the plot for $2 \leq x \leq 4$



is the plot for $3 < x < 6$



is the plot for $-1 < x \leq 7$



We have learned the following skills in this module:

- To solve basic inequalities,
- To flip the inequality sign, or not, according to the rules of algebra,
- To convert several different word problems into inequalities, and
- To graph single-variable inequalities on the number line.
- We also learned the grave consequences of dividing by a (not necessarily positive) variable, and the pitfall of comparing two quantities that have different units.
- We saw that simple questions like

$$x^6 - 42x^5 + 683x^4 - 5404x^3 + 21,303x^2 - 37,562x + 23,021 \leq 2000$$

  often have fairly complicated answers.
- We briefly mentioned the vocabulary terms: *industrial engineering*, *linear programming*, and *operations research*.