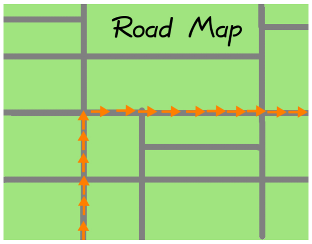
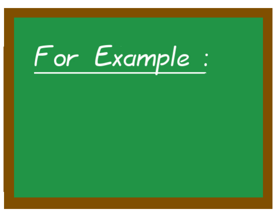
Module 2.11: Cost-Benefit Analysis



A "cost-benefit analysis" problem is a classic style of problem in business. One does cost-benefit analysis to keep costs as low as possible. These problems are done in three steps. First, you find functions—usually very simple ones—which describe each supplier. Second, you solve some inequalities that tell you when one supplier is cheaper than the other. Third, you turn this mathematical observation into a purchasing policy statement.

This topic is unusual in mathematics, because it is a word problem whose answer is not a number, nor a formula, but rather, a policy.



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We're going to start with an off-topic example, to show how inequalities can be used in a word problem. Jennifer is a plastics salesman and is thinking of changing companies. After doing interviews, she gets two offers. Company Alpha will offer \$ 81,000 plus 5% of all sales. Company Beta wishes to inspire its employees to sell more, and so pays them 8% of sales, but no salary whatsoever. What rule should she use to pick the job offer, on the basis of anticipated sales?

Step One: We do not know how much Jennifer will sell each year—it is unknown. Accordingly, let's say that she will sell x dollars per year. Company Alpha will pay her according to the function:

$$a(x) = 81,000 + 0.05x$$

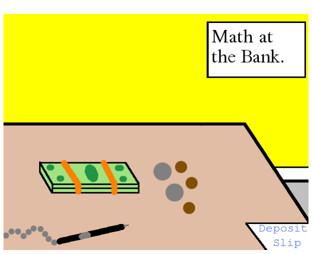
while Company Beta will pay her according to the function:

$$b(x) = 0.08x$$

Step Two: Now let's compute the circumstances when Company Alpha pays more.

$$\begin{aligned} a(x) &> b(x) \\ 81,000 + 0.05x &> 0.08x \\ 81,000 &> 0.08x - 0.05x \\ 81,000 &> 0.03x \\ 81,000/0.03 &> x \\ 2,700,000 &> x \\ x &< 2,700,000 \end{aligned}$$

We will continue the example in the next box.



The last line translates as "x is less than 2.7 millions dollars." This also means that the entire series of mathematical inequality steps translates into "Company Alpha pays better when x is less than 2.7 million dollars."

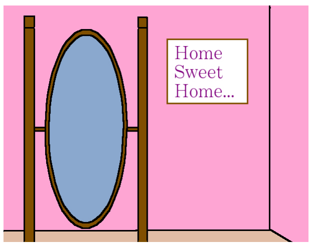
The policy statement would be

- If Jennifer predicts she will sell less than \$ 2,700,000, choose Company Alpha.
- If Jennifer predicts she will sell more than \$ 2,700,000, choose Company Beta.

Note, this does not tell us anything about the case when she sells *exactly* 2.7 million dollars to the penny. In this situation, it turns out that Company Alpha and Company Beta will pay her exactly the same amount, to the penny. The technical term for this is the *ambivalence point*, and that is how we check our work, as shown below.

$$\begin{aligned} a(2,700,000) &= 81,000 + 0.05(2,700,000) = 81,000 + 135,000 = 216,000 \\ b(2,700,000) &= 0.08(2,700,000) = 216,000 \end{aligned}$$

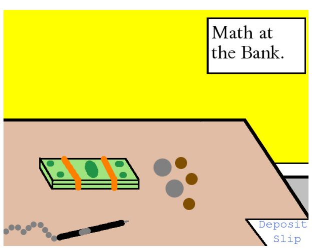
That's great news, because they match exactly, to the penny!



A Pause for Reflection...

Performing the check with the ambivalence point is easy to do. However, it is slightly more complex to understand why it works. Consider when the word "ambivalent" is used outside of mathematics. When you present someone with two options, and that person says "I am ambivalent" it means that both options are equally good or equally bad for them.

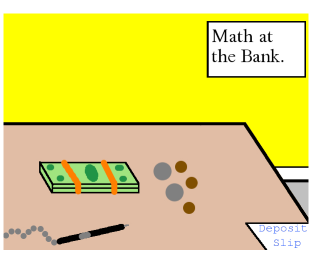
If all the possible amounts of total sales for Jennifer that are less than \$ 2,700,000 cause her to choose Company A, and all the possible amounts that are more than \$ 2,700,000 cause her to choose Company B, then surely it should make sense that at *exactly* \$ 2,700,000 she would be ambivalent about the two companies. Why is she ambivalent in this case? Because she will take home \$ 216,000 in either case.



Before we continue with the module, I'd like to caution you about something. Offers like "Company Beta" above, where you have no salary but get a hefty commission, are actually not uncommon. A very good friend of mine got such an offer (to sell insurance) right after graduating college. He wasn't really cut out for sales but he didn't know that ahead of time.

In any case, after working hard for six months, he had not sold even one dollar of insurance. Accordingly, he did not receive even one penny from the firm. Naturally, he also had living expenses during this six-month time frame, and so he acquired some more debt. Luckily, his parents were wealthy and so he didn't end up destroying his credit rating or becoming homeless, but one could imagine that someone of more ordinary circumstances would have been very significantly set back by this experience.

In the end, he went to get a graduate degree in something that matched his personality better, and is now perfectly happy with his new career choice, but nonetheless this was an unpleasant experience for him.

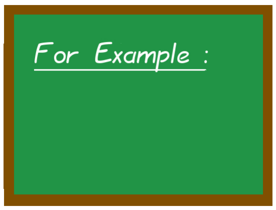


Now let's look at the situation of the previous box from the perspective of the insurance company.

Why do companies like giving such offers to those newly graduated from college? Because some new college graduates will not succeed at their first job (just like my friend) and in that case, the company has not lost any money for salary. If they had paid my friend wages, then the insurance company would have lost a lot of money. For example, if he had a \$ 41,000 salary, in six months, they'd have paid him \$ 20,500 plus overhead.

As a result, companies like giving offers of this nature so that hiring some inexperienced young people will entail less financial risk.

Now that we've warmed up with a business-related inequalities problem, (and we've been warned about the dangers of accepting commission-only sales jobs), we can begin our study of cost-benefit analysis problems with a typical example.



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Company A sells stationary at the following rate: \$ 1000 for the first 100 reams, and then \$ 8 for each additional ream. Meanwhile Company B sells stationary for \$ 9 per ream. You know for sure you'll use more than 100 reams, but you aren't sure how much. What rule should be used for deciding which company to select for each order?

Let n be the number of reams required. First, we can write $b(n) = 9n$, to signify that each ream from Company B costs \$ 9. For Company A, we know that we'll be using more than 100 reams, so we'll pay \$ 1000 for the first 100 reams, and order $n - 100$ at \$ 8 each. That means

$$a(n) = 8(n - 100) + 1000$$

which can be simplified to

$$a(n) = 8n + 200$$

With these two functions, we can proceed to write inequalities. Let's compute when Company B is cheaper than Company A.

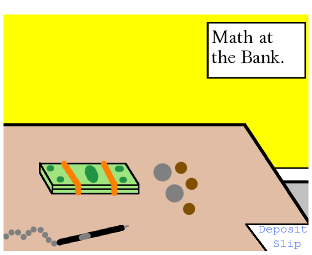
$$\begin{aligned} b(n) &< a(n) \\ 9n &< 8n + 200 \\ 9n - 8n &< 200 \\ n &< 200 \end{aligned}$$

Therefore, we learn that if $n < 200$ then Company B is cheaper. This would also imply that if $n > 200$ then Company A is cheaper. The final policy would be "If ordering less than 200 reams, use Company B; if ordering more than 200 reams, use Company A."

We can check our work from the previous box by plugging in the ambivalence point, $n = 200$, into each of the functions.

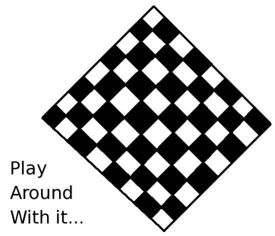
$$\begin{aligned} a(200) &= 8(200) + 200 = 1600 + 200 = 1800 \\ b(200) &= 9(200) = 1800 \end{aligned}$$

Because we computed the same answer for each one, we know that we found the right ambivalence point. The only remaining thing is to check to see if we perhaps got Company A and Company B backwards—sometimes students do that. We can check that visually, and then rest assured that we got the problem correct.



To further explore the situation of the last two boxes, let's imagine that Andrew, Bob and Charlene are each managers of divisions inside a large corporation—divisions that use lots of paper. Bob uses Company B all the time, and Andrew uses Company A all the time. Meanwhile, Charlene has taken this course, and so she computes the same policy that we just computed in the last two boxes. In various months, the divisions might make very differently sized orders depending on their needs. Let's see how much Andrew, Bob and Charlene are going to spend:

	Andrew Company A	Bob Company B	Charlene by policy
140	\$ 1320	\$ 1260	\$ 1260
160	\$ 1480	\$ 1440	\$ 1440
180	\$ 1640	\$ 1620	\$ 1620
200	\$ 1800	\$ 1800	\$ 1800
220	\$ 1960	\$ 1980	\$ 1960
240	\$ 2120	\$ 2160	\$ 2120
260	\$ 2280	\$ 2340	\$ 2280



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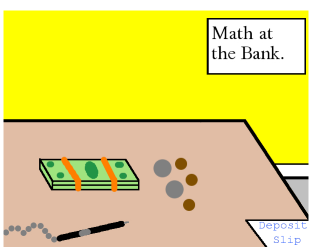
Continuing with the previous box, let's look at how much Charlene is saving, by using a policy to choose a different paper supplier for each purchase, as compared to what Andrew and Bob are doing.

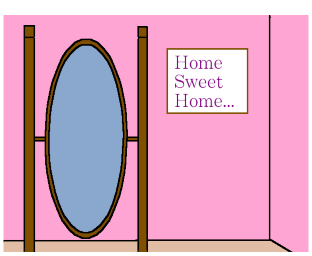
- Perhaps in December, when business is slow for the holidays, 140 reams are needed. How much, as a percentage, is Charlene's division saving as compared to Andrew's division? [Answer: Charlene is saving 4.54%.]
- Perhaps in April, when business is booming, 260 reams are needed. How much, as a percentage, is Charlene's division saving as compared to Bob's division? [Answer: Charlene is saving 2.56%.]

You might want to recall the $ 654,321-example from the preface (see Page 13), where we showed the enormous impact of making merely five cost-saving changes that each come to a mere 2%. In the previous box, we see that Charlene has done better than that.



Cost-Benefit Analysis as a topic is fairly old, and I have seen problems of this type in mathematics books from the 1960s. However, the topic gained renewed popularity in the age of email and the web.

All Charlene need do is write an email with her one or two sentence policy, and then list the contact data for the two suppliers. This way, when anyone needs to order paper, they can just put in the name of *either* supplier into the search box of their email tool. Then in a single email, they have their guideline for making a decision, and all the contact data. It costs *nothing* to implement a policy of this type.



A Pause for Reflection...

Let's take a few minutes to reflect on the last few boxes. Common sense, or business intuition, would normally cause someone to say "Either Company A or Company B is right for me, and I just have to figure out which one." Here, however, we see that common sense was just wrong.

Using one company or the other for each order, depending on the size of the order, was Charlene's strategy and it is more efficient. Her expenses were, for any size order, always less-than-or-equal-to either Andrews or Bob's.

As we saw in the previous box, common sense is often wrong. This is just one of many examples where one must never follow intuition, but instead, one should engage in computation.

For Example :

2-11-4

You are the director of a factory, and a new gadget is going to be manufactured there. You send out proposals to machine-tool dealers, and two bids come back. From Company A, the machine costs $ 300,000 and will manufacture the gadget at a cost of $ 20 each. From Company B, the machine costs $ 100,000 and will manufacture the gadget at a cost of $ 40 each. Naturally, if you are planning to manufacture many items, then Machine A makes sense; if you are planning to manufacture very few, then Machine B makes sense. You're going to email the marketing department to get an estimate on the number of gadgets that are anticipated to sell, and when the answer arrives, what rule will you use to decide which machine to buy?

The function for Machine A is given by

$$a(n) = 20n + 300,000$$

while the function for Machine B is given by

$$b(n) = 40n + 100,000$$

which allows us to determine which values of n will make Company A's machine cheaper. The computation is in the next box.

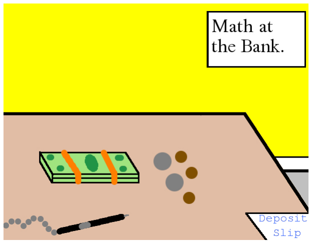
Continuing with the previous box, we have

$$\begin{aligned} a(n) &< b(n) \\ 20n + 300,000 &< 40n + 100,000 \\ 20n + 200,000 &< 40n \\ 200,000 &< 20n \\ 10,000 &< n \end{aligned}$$

Therefore, we learn that Machine A is cheaper if $n > 10,000$. This means that Machine B is cheaper if $n < 10,000$. If the marketing department anticipates a demand of more than 10,000 gadgets, purchase Machine A; if the marketing department anticipates a demand of fewer than 10,000 gadgets, purchase Machine B.

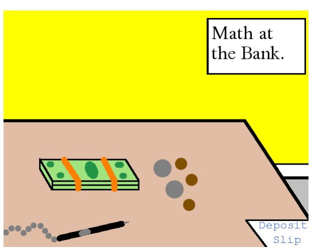
If we assume a sales price of $ 80 per gadget, then we would have the following cost, revenue, and profit structure for the previous problem. Take a moment to verify one of the rows, perhaps 8000 gadgets, before continuing onward.

n	Cost Machine A	Revenue Machine A	Profit Machine A		Cost Machine B	Revenue Machine B	Profit Machine B	
2000	$ 340,000.00	$ 160,000.00	-$ 180,000.00	LOSS!	$ 180,000.00	$ 160,000.00	-$ 20,000.00	LOSS!
4000	$ 380,000.00	$ 320,000.00	-$ 60,000.00	LOSS!	$ 260,000.00	$ 320,000.00	$ 60,000.00	profit
6000	$ 420,000.00	$ 480,000.00	$ 60,000.00	profit	$ 340,000.00	$ 480,000.00	$ 140,000.00	profit
8000	$ 460,000.00	$ 640,000.00	$ 180,000.00	profit	$ 420,000.00	$ 640,000.00	$ 220,000.00	profit
10,000	$ 500,000.00	$ 800,000.00	$ 300,000.00	profit	$ 500,000.00	$ 800,000.00	$ 300,000.00	profit
12,000	$ 540,000.00	$ 960,000.00	$ 420,000.00	profit	$ 580,000.00	$ 960,000.00	$ 380,000.00	profit
14,000	$ 580,000.00	$ 1,120,000.00	$ 540,000.00	profit	$ 660,000.00	$ 1,120,000.00	$ 460,000.00	profit
16,000	$ 620,000.00	$ 1,280,000.00	$ 660,000.00	profit	$ 740,000.00	$ 1,280,000.00	$ 540,000.00	profit
18,000	$ 660,000.00	$ 1,440,000.00	$ 780,000.00	profit	$ 820,000.00	$ 1,440,000.00	$ 620,000.00	profit
20,000	$ 700,000.00	$ 1,600,000.00	$ 900,000.00	profit	$ 900,000.00	$ 1,600,000.00	$ 700,000.00	profit



Is the comparison in the previous box not shocking? There are several rows of the table where making the wrong choice of machine can be catastrophic for your business.

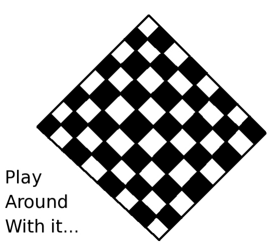
- If you choose Machine B and only 2000 gadgets are ordered, then your company loses \$ 20,000. This can be survived, and ascribed to bad luck. However, if you choose Machine A and only 2000 gadgets are ordered, then your company loses \$ 180,000. Not only are you certain to be fired, they'll probably sue you as well.
- If you choose Machine A and only 6000 gadgets are ordered, then your company makes a \$ 60,000 profit—which is nice. However, had you chosen Machine B then they would have made \$ 140,000 profit. So you must hope that no one in the corporation ever happens to notice this missed opportunity.
- If 10,000 gadgets are ordered, then the profit is \$ 300,000 regardless which machine you choose to purchase. That's kind of interesting.
- If 20,000 gadgets are ordered, then the profit with Machine B is \$ 700,000—a major achievement. However, if Machine A is used, then the profit is \$ 900,000—even better!
- By the way, you really should take a moment to see if you can reproduce one of the lines of the above table. It could be an exam question.



Please be sure to write the policy as a complete and proper sentence (or two), as I have done in the answers up to this point. For example, we've already seen in this module the following policies:

- If Jennifer predicts she will sell less than \$ 2,700,000, choose Company Alpha; if Jennifer predicts she will sell more than \$ 2,700,000, choose Company Beta.
- If ordering less than 200 reams, use Company B; if ordering more than 200 reams, use Company A.
- If the marketing department anticipates a demand of more than 10,000 gadgets, purchase Company A's machine; if the marketing department anticipates a demand of fewer than 10,000 gadgets, purchase Company B's machine.

Notice, we leave off the case of exactly 2.7 million dollars in sales, ordering exactly 200 reams of paper, or a demand of exactly 10,000 reams. That's because both options cost the same at those values, which are the ambivalent points of each problem.



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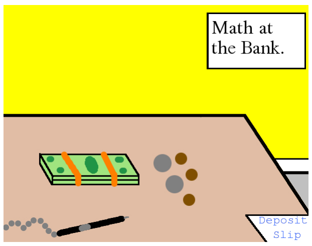
A social worker at the local hospital helps patients who are the victims of accidents find the right law firm. There is a law firm that charges a fee of \$ 10,000 plus 25% of the verdict in civil cases (call them "Firm A"). Another law firm charges nothing upfront but takes one-third of the verdict (call them "Firm B"). What rule will help the plaintiff choose the law firm of minimum cost, *on the basis of this information alone.*

[Answer: if the verdict is anticipated to be greater than \$ 120,000 then use "Firm A," but if the verdict is anticipated to be less than \$ 120,000 then use "Firm B."]

Just as before, we can check our work by plugging in the ambivalence point into both functions, and hope that they come out the same.

$$\begin{aligned} a(120,000) &= 10,000 + 0.25(120,000) = 10,000 + 30,000 = 40,000 \\ b(120,000) &= (1/3)(120,000) = 40,000 \end{aligned}$$

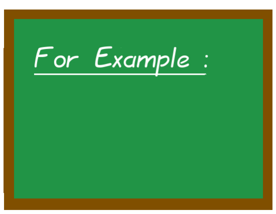
As you can see, it is perfect match.



A few businesses use the "we only get paid if you get paid" strategy. This means that the law firm will not receive any revenue at all from a case, unless they win in court or extract a cash settlement from the defendants. Likewise, writers of fiction novels have literary agents who use this model. The literary agent only gets paid, at all, if they find a publisher willing to publish the author's new novel. Realtors for selling a home use this model too, where they only receive revenue at all when the house is successfully sold.

Some consumers find it easier to trust that law firms, literary agents, and realtors who use this model are going to actually work hard toward achieving a successful outcome. It was for this reason that we said "on the basis of this information alone" in the previous checkerboard box. However, other consumers might prefer to use ratings or other measures to select a law firm, literary agent, or realtor.

The next example has a useful tip for students who cannot decide if "Firm A" or "Firm B" should get the small verdicts. If you find yourself getting the two companies or suppliers interchanged when doing these problems, this short cut will help.



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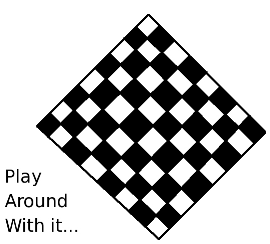
Just plug some number, not equal to the ambivalence point, into both functions. In the above example, we could select \$ 60,000. We would get for Firm A

$$a(60,000) = 10,000 + 0.25(60,000) = 25,000$$

and likewise we would get for Firm B

$$b(60,000) = (1/3)(60,000) = 20,000$$

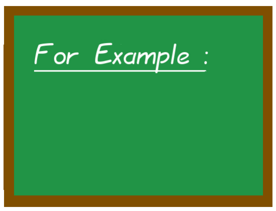
and so it is clear for the \$ 60,000 verdict, that Firm B is the better choice. This means that all lawsuits with verdicts less than \$ 60,000 should use Firm A. Accordingly, all lawsuits with verdicts more than \$ 60,000 should use Firm B.



2-11-7

Going back to the example of Company A and Company B on Page 388, let's use the tip from the previous box to help decide which company should get the small paper orders, and which should get the largest. We had $a(n) = 8(n - 100) + 1000$ and $b(n) = 9n$. The ambivalence point was $n = 200$ so let's try $n = 300$.

- What is $a(300)$? [Answer: \$ 2600.]
- What is $b(300)$? [Answer: \$ 2700.]
- Which company is cheaper for 300 reams? [Answer: Company A.]
- Which company is cheaper for all orders above 200 reams? [Answer: Company A.]
- Which company is cheaper for all orders below 200 reams? [Answer: Company B.]



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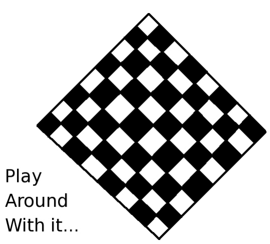
Two competing websites are offering movies for download. One costs \$ 27.95 per month, for unlimited access, and the other costs \$ 5 per movie, but no monthly fee. Which one should you get? Obviously if you watch 10 movies per month, then you'd rather pay \$ 27.95 than \$ 50; if you watch 1 movie per month, then you'd rather pay \$ 5 than \$ 27.95. What rule should help you decide which plan you should take?

By now, you can calculate yourself that the functions $a(n) = 27.95$ and $b(n) = 5n$. Then we would compute when the unlimited service, $a(n)$, was cheaper.

$$\begin{aligned} a(n) &< b(n) \\ 27.95 &< 5n \\ 5.59 &< n \end{aligned}$$

We learn that the inequality $a(n) < b(n)$, which can also be written as $27.95 < 5n$ will be satisfied whenever $n > 5.59$, but will be dissatisfied whenever $n \leq 5.59$. Nonetheless, this does not make sense in terms of the original problem, because you cannot rent 5.59 movies, or any non-integer number of movies.

Therefore, the 5.59 signals you to use two regions. The first region is "5 or fewer" and that represents the inequality being unsatisfied, or $b(n)$ being cheaper. The second region is "6 or more" and that represents the inequality being satisfied, or $a(n)$ being cheaper.



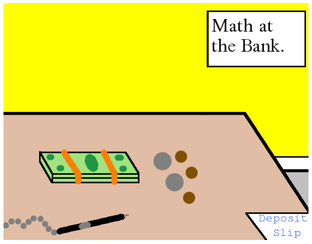
2-11-9

While most checking accounts in the USA have done away with annual fees and other fees for each check, business checking accounts often still have fees. Let us consider two banks for a business checking account.

First, Bank A has a \$ 24 per year annual fee, but ordering 200 checks costs \$ 20. Meanwhile, Bank B has a \$ 50 per year annual fee, but ordering 200 checks costs \$ 10. If someone in a local office of the Small Business Administration (`www.sba.gov`) were to be advising new entrepreneurs, we should try to determine what policy should guide their advice. Each founder of a new business would first estimate how many checks they estimate that they will write per year.

- How much does each check cost with Bank A? [Answer: 10 cents per check.]
- How much does each check cost with Bank B? [Answer: 5 cents per check.]
- What inequality will reveal when Bank A is cheaper? [Answer: $24+0.1n < 50+0.05n$.]
- What is the final policy?

[Answer: if the business will write fewer than 520 checks per year, then they should select Bank A. However, if the business will write more than 520 checks per year, then they should select Bank B.]

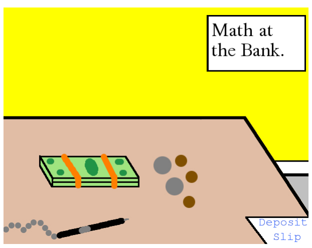


Earlier, when we looked at machining gadgets on Page 390, we saw that making the wrong choice can be catastrophic. It turns out that this was because we were looking at cases far from the ambivalence point. The flip side of that is when you are close to the ambivalence point, there is almost no consequence to making the wrong choice.

To illustrate this point, we can examine the business checking account problem from the previous box.

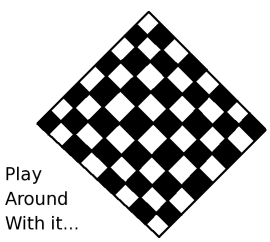
- If $n = 515$ then $a(515) = 75.50$ while $b(515) = 75.75$. This is not an earth-shattering difference. It is a gap of 25 cents.
- Similarly, if $n = 525$ then $a(525) = 76.50$ while $b(525) = 76.25$. Again, not a huge deal, but merely 25 cents.

So for these values, very close to the ambivalence point, we see that there is not a large difference in the costs for the two options. In business, this means that you do not have to agonize over the decision when near the ambivalence point. However, far from the ambivalence point, making the wrong choice could be catastrophic to your career path, as we have already seen in the gadget-manufacturing problem.



The concept of *pro rata* versus *or fraction thereof* was explained on Page 170 in the module "Working with Functions." If you already know or remember what that means, feel free to continue to the next box. If you have no idea what that means, flip back to that module and read about it.

For those in between, who'd like a quick reminder: if a shipping company charges \$ 6 per pound, *pro rata*, that means that a package which weighs 1.5 pounds will cost \$ 9.00 to ship, because $6 \times 1.5 = 9$. Simiarly, a package that weighs 1.1 pounds will cost \$ 6.60 to ship. However, if they charge \$ 6 per pound, *or fraction thereof*, that means that a package which weighs 1.5 pounds or a package that weighs 1.1 pounds will cost the same to ship as a two pound package—namely \$ 12.00.

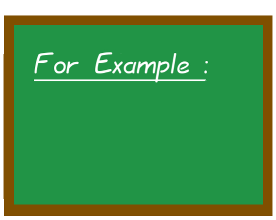


2-11-10

Sending express mail with one service costs \$ 15.95 for the first pound, and \$ 2.95 for each additional pound, *pro rata*. Call this Company C.

Meanwhile, a competing service offers \$ 4.95 per pound, *pro rata*, and you may call this Company D. What rule should you use when mailing a package to keep the cost low?

[Answer: if the package weights less than 6.5 pounds, then send it via Company D; however if the package weights more than 6.5 pounds, then send it via Company C.]



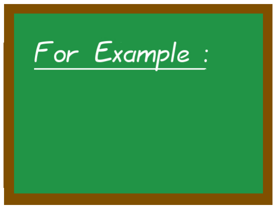
2-11-11

In the previous box, we considered a pair of shipping companies using a *pro rata* system of charging their costumers. What would happen if they were using *or fraction thereof* instead of *pro rata*?

We know from two boxes ago that packages with weight 6 pounds or less are definitely better with Company D, because 6 pounds is less than the ambivalence point of 6.5 pounds. However, even 6.0001 pounds would be treated as 7 pounds, and we know 7-pound packages would be better shipped with Company C.

Therefore, packages weighing 6 pounds or less should be shipped with Company D. Packages weighing more than 6 pounds should be shipped with Company C.

Overall, the difference is usually small between the answer when solving a problem *pro rata* or with the *or fraction thereof* method. Therefore, we will not focus upon *or fraction thereof* problems.



2-11-12

A dental clinic gives out free tooth brushes to all their patients who are under 25. Therefore, they need lots of tooth brushes. Company A sells them for 6 dollars per box of a dozen, pro rata, but there is a \$ 100 per order delivery charge. Company B sells them for 39 dollars per box of a hundred, pro rata, but there is a \$ 232 per order delivery charge. What rule should help the clinic decide which company to order from depending on the size of their order?

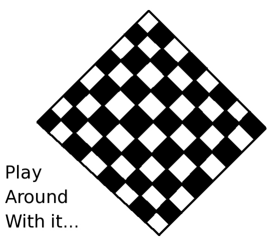
First, let's consider Company A. If we need n toothbrushes, that's going to be $n/12$ boxes of a dozen, costing us $6(n/12)$ dollars. Then, we need the per-order-delivery charge of \$ 100. Therefore, we write

$$a(n) = 100 + 6(n/12) = 100 + n/2$$

Similarly, let's consider Company B. If we need n toothbrushes, that's going to be $n/100$ boxes of a hundred, costing us $39(n/100)$ dollars. Of course, we must add the per-order-delivery charge of \$ 232. Therefore, we write

$$b(n) = 232 + 39(n/100) = 232 + 0.39n$$

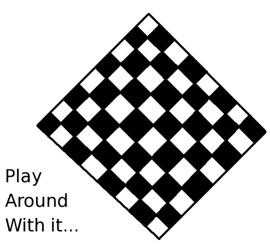
You will now complete the rest of the problem yourself, as an exercise.



2-11-13

Continuing with the previous box...

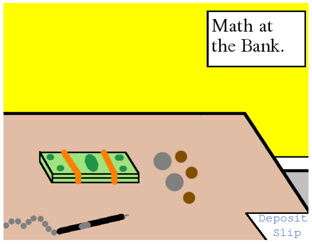
- For what values of n is $a(n) < b(n)$? [Answer: $n < 1200$.]
- What is the ambivalence point? [Answer: 1200 toothbrushes.]
- If the office requires 600 toothbrushes, what is the cost with Company A? [Answer: \$ 400.]
- If the office requires 600 toothbrushes, what is the cost with Company B? [Answer: \$ 466.]
- If the office requires 2400 toothbrushes, what is the cost with Company A? [Answer: \$ 1300.]
- If the office requires 2400 toothbrushes, what is the cost with Company B? [Answer: \$ 1168.]
- What is the policy that will lead to the cheapest purchase each time?
  [Answer: If less than 1200 toothbrushes are required, then use Company A. If more than 1200 toothbrushes are required, then use Company B.]



2-11-14

Rental Car Company A charges \$ 45 plus 30 cents per mile, but Rental Car Company B charges \$ 60 plus 10 cents per mile. What policy will decide what company should be used on a trip, based on the mileage travelled?

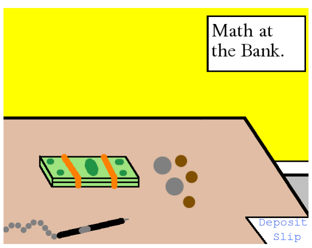
- What function represents the cost of Company A? [Answer: $a(n) = 45 + 0.3n$.]
- What function represents the cost of Company B? [Answer: $b(n) = 60 + 0.1n$.]
- What is the final policy statement? [Answer: If the trip is to be less than 75 miles, use Company A. If the trip is to be more than 75 miles, use Company B.]



Let us imagine Andrew uses only Company A, and Bob uses only Company B. On the other hand, Charlene uses the policy devised in the previous box. Here is how much the three of them would pay for various lengths of driving trip.

	Andrew Company A	Bob Company B	Charlene by policy
20 miles	$ 51	$ 62	$ 51
40 miles	$ 57	$ 64	$ 57
60 miles	$ 63	$ 66	$ 63
80 miles	$ 69	$ 68	$ 68
100 miles	$ 75	$ 70	$ 70
120 miles	$ 81	$ 72	$ 72

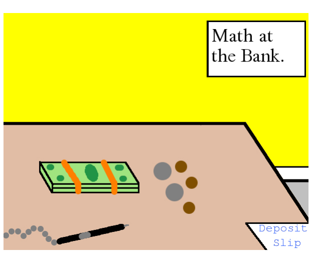
This is pretty substantial. Charlene's plan is never more expensive than either Andrew or Bob. Yet there's several trips where Andrew is more expensive than Charlene, and none where he is less expensive than Charlene. So for sure, Andrew is going to perform worse over the sum of lots of trips than Charlene—likewise with Bob.



Let's take this opportunity to verify our notion that choosing the wrong company near the ambivalence point is not a big deal—but choosing the wrong company far from the ambivalence point often is very noticeable. We will use the functions from the previous checkerboard.

- if $n = 73$ miles, then $a(n) = 66.90$ and $b(n) = 67.30$.
- if $n = 77$ miles, then $a(n) = 68.10$ and $b(n) = 67.70$.
- As you can see, it is not a large difference. In the 73-mile case it comes to 40 cents difference, and in the 77-mile case, it comes to 40 cents difference as well.

We will continue our analysis in the next box.



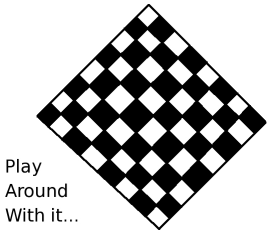
Continuing with the previous box, we've confirmed that making the wrong choice near the ambivalence point has rather minor consequences. Let's now confirm that making the error far from the ambivalence point has major consequences.

- if $n = 15$ miles, then $a(n) = 49.50$ while $b(n) = 61.50$.
- Here $b(n)$ represents an overpayment of $ 12 or 24.24%.
- if $n = 150$ miles, then $a(n) = 90.00$ while $b(n) = 75.00$.
- We can see that $a(n)$ represents an overpayment of $ 15 or 20%.

When we studied the 654,321-example, we saw that savings of 2% can mean the difference between profit and loss. Here, the percentage savings is in the 20s! If there are 10 employees renting cars, 5 days a week, 50 weeks per year, each for trips of 150 miles, with cars rented from Company A, then that's a waste of

$$10 \times 5 \times 50 \times 15 = \$\ 37,500$$

which certainly no senior manager would ever forgive.

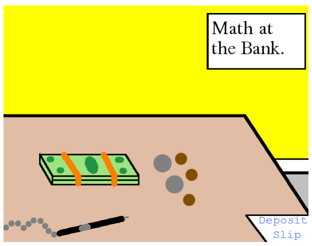


2-11-15

Perhaps a good friend of yours is going on a study abroad program and you want to keep in touch. Your current international calling rate on your land line is 10 cents per minute. You see an advertisement for long-distance cards that charge 1.5 cents per minute. However, you read the fine print on the back of the long distance card and you discover that there is a 69 cent "connection fee" per phone call.

What rule should you decide when dialing, to choose between either using the card or using your regular service? (Assume the pro rata system of billing.)

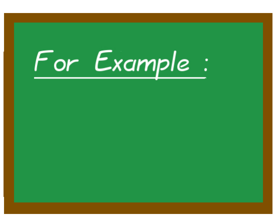
[Answer: if the call is to be shorter than $8.11764\cdots$ minutes, use the land line, but if the call is to be longer than $8.11764\cdots$ minutes, use the calling card.]



Does the policy in the previous box look a little strange? The "4" in 8.11764 is representing $4/100,000$ minutes, or 0.0024 seconds. That's between 1/417th and 1/416th of a second. For sure, we can cut that digit off.

Now, what it comes down to is that in all likelihood, the phone company rounds to the nearest second. So we want to write $8.11764\cdots$ in terms of seconds. We learned how to do this on Page 83 during the module "Order of Operations and Calculator Skills." We know we need 8 minutes, and $60 \times 0.11764\cdots = 7.0588\cdots$ so we would write 8 minutes and 7 seconds.

So far, all the functions that we've needed have been linear—sometimes called "degree one." You might be wondering if quadratic functions ever come up in cost-benefit analysis. It is rare but possible. We'll see an example of that now.



2-11-16

You are having all of your office spaces re-carpeted, and get bids from the local suppliers. Carpeting Company C is offering free installation, and charges \$ 2.75 per square foot. The other company, R, is offering \$ 2.25 per square foot, but charges a \$ 72 per room installation fee. Your offices are mostly $10' \times 10'$ or $20' \times 20'$, but some are $15' \times 15'$ and there are other sized rooms too. For simplicity, assume every room is a square. What rule should be used for each room to determine which supplier will carpet it?

Of course, it goes without saying that an $s \times s$ room has s^2 square feet to carpet. Thus for Company C we have

$$c(s) = 2.75s^2$$

whereas for Company R we have

$$r(s) = 72 + 2.25s^2$$

Now let's find the values of s such that Company C is cheaper.

$$\begin{aligned} c(s) &< r(s) \\ 2.75s^2 &< 72 + 2.25s^2 \\ 2.75s^2 - 2.25s^2 &< 72 \\ 0.5s^2 &< 72 \\ s^2 &< 144 \end{aligned}$$

We will pause here to discuss a theoretical issue.

When one sees $s^2 < 144$, it is very tempting to just write $s < 12$ on the next line. You might be wondering if this legal or illegal.

It turns out that it is both. In this problem, the value of s cannot be negative, nor can it be zero. Thus, s is only positive. (After all, it is hard to imagine an office that is smaller than $3' \times 3'$.)

When all the variables of an inequality are known to be positive, you can square root both sides. In general, however, square rooting both sides of an inequality is illegal, and will produce a wrong answer. We will explore this question in more detail in the chapter called "Inequalities and Industrial Engineering."

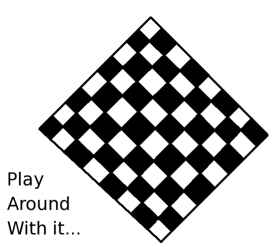
In general, it is not the case that $s^2 < 144$ is identical to $s < 12$. Consider the case of $s = -20$.

- Is $-20 < 12$? Yes! So we know that -20 satisfies $s < 12$.
- Is $(-20)^2 < 144$? Because $(-20)^2 = 400$, the answer is no—surely 400 is not smaller than 144. Thus we know that -20 does not satisfy $s^2 < 144$.
- Because $s < 12$ and $s^2 < 144$ disagree on the status of $s = -20$, we know that $s < 12$ and $s^2 < 144$ are not synonyms.

In conclusion, you should not square root both sides of an inequality unless you know that both sides are surely positive. In this case, we knew that s was positive, and likewise we see that 144 is positive, so the maneuver was legal.

Now we will continue with the previous example, where we left off. We had $s^2 < 144$, and for this problem only, we permit ourselves to write $s < 12$. Therefore, our final policy would be

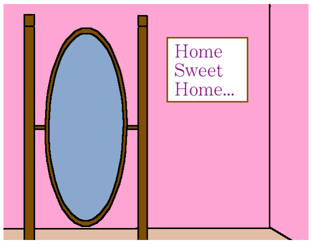
- For offices that are smaller than 12×12, use Company C.
- For offices that are larger than 12×12, use Company R.



2-11-17

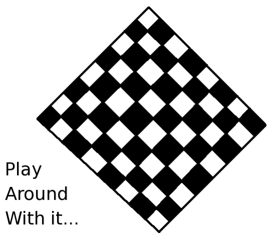
One country club near you, The Alpine Club, has an annual fee of $ 400, and each 9-hole game costs $ 19. Another country club near you, The Bovine Club, has an annual fee of $ 700, and each 9-hole game costs $ 15. If you were to choose your country club solely on the basis of the total annual cost of playing g games of golf, then how would make your decision? (Assume that you are too busy to ever consider playing an 18-hole game.)

[Answer: if you're going to play 74 or fewer games per year ($g < 75$), then choose The Alpine Club; if you're going to play 76 or more games per year ($g > 75$), then choose The Bovine Club.]



A Pause for Reflection...
Just like the case of the personal-injury lawyer, the literary agent, and the realtor, you might not want to decide solely on the basis of the total cost of playing golf. Consider someone who plays a dozen games per year. The Alpine Club costs $ 628 per year and The Bovine Club $ 880 per year. If you're using the club to meet business contacts, it depends what you are in the business of.

For example, if you're helping senior citizens choose medicare options, or if you're selling discount insurance, then you'd probably be more successful circulating yourself among frugal people. However, if you're selling yachts or engaged in private investment banking (wealth management activities), then your target audience would be people who have more money than they know what to do with. Naturally, non-business considerations, such as location, club culture and the quality of the wine list might be factors too.



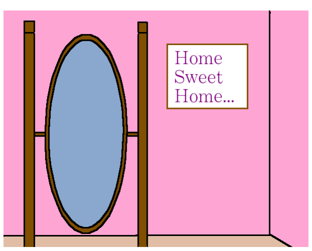
2-11-18

Consider the problem from Page 173 where we are choosing from printers with the following attributes:

- Printer B costs $ 169 and a page of photographs is quoted as being 41 cents.
- Printer C costs $ 499 and a page of photographs is quoted as being 33 cents.

Find out what policy should assign which printer to each office, given that over the lifetime of the printer they will print n pages (single-sided).

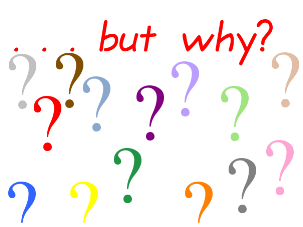
[Answer: if $n < 4125$ then choose Printer B, but if $n > 4125$ then choose Printer C.]



A Pause for Reflection...
Now consider that you're a new manager of a department in a division of three departments and many employees. One fellow manager is very frugal, and wants to always buy the cheapest printer, and wants to give each employee Printer B. Another manager is very far-sighted, long-term oriented, and says that Printer C has lower per-page cost, and so will always cost the company less in the long run.

At this point in the module, you know that both of your more experienced colleagues are wrong. They are using intuition instead of calculation. Think how you might explain to them that it is better to consider which printer will be better for each employee on a case-by-case basis, based upon how many pages they will print over the lifetime of the printer.

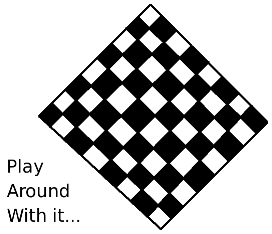
Often in your business life you will be right and your boss will be wrong. It is critical to learn how to tell someone that and still remain on good terms with them.



In the original printer problem from Page 173, there were four printers plus the local copy center. Here, each and every one of our problems deals with two suppliers.

In real life, there might be several possible suppliers. Therefore, you might wonder how to do cost-benefit analysis with multiple suppliers. We'll study this later in the chapter called "Inequalities and Industrial Engineering."

The following problem is inspired by a problem which appeared in *College Mathematics—for Business, Economics, Life Sciences, and Social Sciences*, by Barnett, Ziegler, and Byleen, 11th edition. It was Problem 1-1-62.

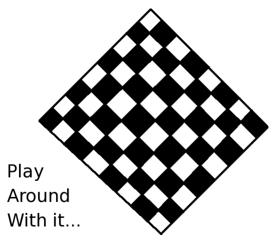


2-11-19

Pat is opening a floor-waxing business for the Empire State Building. Conveniently, each floor has very nearly the same surface area. Pat is trying to decide on renting a commercial waxing machine or a (larger) industrial one. He can rent the commercial machine for $ 299 per day, but it takes expensive wax that costs $ 240 per bottle, and needs about 2 bottles per floor. Alternatively, for $ 4499 per day, he can rent an industrial machine from a janitorial-supply company. There, he can get cheap wax that is $ 141 per bottle, but that requires 3 bottles per floor. Which machine should he buy, based on the number of floors he thinks will choose to hire him?

- How much money does each floor cost with the commercial machine?
  [Answer: $ 480 per floor.]
- How much money does each floor cost with the industrial machine?
  [Answer: $ 423 per floor.]

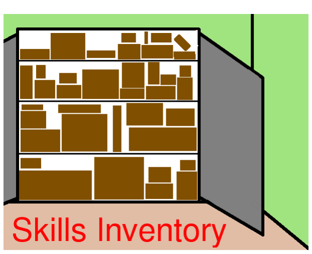
As you can see, the industrial machine only makes sense if Pat's going to wax 74 or more floors. On the other hand, if Pat is going to wax 73 or fewer floors, then the commercial machine make sense.



2-11-20

Repeat the above problem if the cheap wax falls in price to $ 99 per bottle. Just determine the final policy.

[Answer: Now the commercial machine is better for 22 or fewer floors, whereas the industrial machine is better for 23 or more floors.]



Our exploration of this topic has now concluded. We have learned the following skills in this module:

- to compare job offers mixing salary and commission, as well as to identify the personal risk that comes with a commission-only sales job,
- to use a pair of functions and an inequality to perform cost-benefit analysis,
- to check cost-benefit analysis using the ambivalence point,
- to write the answer to a cost-benefit analysis problem as a one or two sentence policy,
- to adapt this method to situations where only integer answers would make sense,
- as well as the vocabulary terms: *ambivalence point*, "*or fraction thereof*," and *pro rata*.
- We saw that there are many realistic situations where a policy that carefully mixes suppliers (to minimize costs) will out perform a policy of using any one supplier all the time.
- We also saw that the impact of choosing the wrong supplier is small near the ambivalence point, but the impact is large away from it.
- We learned that one must not square root both sides of an inequality, unless one is sure that both sides are positive.