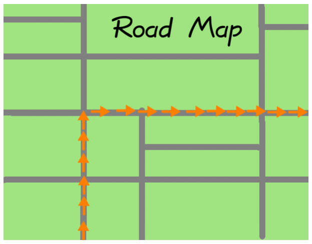
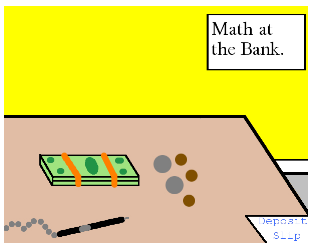
Module 2.2: The Basics of Compound Interest



In this section, we're going to learn about *compound interest*. Understanding compound interest is vital, as it is one of the strongest forces in the financial world. Nearly all financial transactions involving interest use compound interest, and the topic will recur frequently throughout the book.



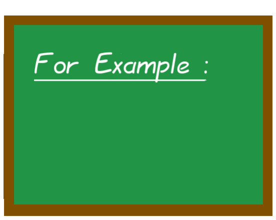
Most of us know what a *checking account* or a *savings account* is. A few checking accounts give interest, while all savings accounts do. Of course, the bank gets its money (for the interest it pays) from lending your savings out to others, in the form of loans, mortgages, and so forth. The problem with savings accounts is that the bank never knows when you're going to come and withdraw your money. Therefore, with 1000s of customers, they have to leave a good chunk of the money uninvested, to be able to pay up when a few people demand a withdrawal on the same day.

In contrast to the types of accounts mentioned in the previous box, with a *certificate of deposit*, you promise not to withdraw any of the money before a fixed day, unless you are willing to pay an early withdrawal penalty. Thus, if you buy a 1-year certificate, you are contracting with your bank to leave this money with them for the entire year. If you really need it, you can withdraw it for a small penalty, but the idea is that you are unlikely to do so because of the penalty. Since the bank benefits from having this extra promise of withdrawals being unlikely, they often give a much higher interest rate—often over double the interest rate of an ordinary savings account.

The slang term for a certificate of deposit is "a *CD*." Finally, note that the duration does not have to be one year. It can be two years, half a year, or a quarter. Rarely, one sees five-year certificates of deposit as well.

A compound interest problem is actually a bunch of simple interest problems glued together, as we are about to explore. The idea is that in a sequence of time periods, your money earns interest, and that money gets re-deposited. In this sense, the interest builds on itself, because the growth of your wealth is being reinvested.

In nearly all circumstances, we solve compound interest problems using various compound interest formulas. However, I think it is very useful to see how compound interest grows out of simple interest. Therefore, just once, we will watch that process in detail.



2-2-1

Suppose you have a friend in his first year of college, perhaps his name is Rich. Imagine that a relative of Rich dies at the start of his freshman year, and leaves him $ 40,000. Rich doesn't have much use for $ 40,000 right now, but perhaps he's going to graduate from college in 4 years, and would like to use the money at that time to buy a nice car. He could get a certificate of deposit for a year, and suppose there is one available for 3% (which is about typical). Recall from the previous module that the formula for simple interest is $A = P(1 + rt)$.

Therefore, after 1 year, Rich has

$$A = (40,000)(1 + 0.03(1)) = (40,000)(1.03) = 41,200$$

However, his certificate of deposit is finished (the banker would say "matured") and accordingly, Rich buys another one. We will continue our analysis in the next box.

Continuing with the previous box, the interest rate would likely have changed during that year, but for sanity's sake, let's imagine that it stays at 3%. Using the simple interest formula again, we get

$$A = (41,200)(1 + 0.03(1)) = (41,200)(1.03) = 42,436$$

That's not bad! Instead of leaving the money in his apartment (perhaps in his sock drawer, where it could get stolen or be destroyed by fire) Rich does the right thing and puts the money in a bank. For that effort (which took 1 hour at most), even at this meager rate of interest, he has earned over \$ 2,400.

After the end of the second year, it is time to buy a third certificate of deposit, and this one is going to be worth

$$A = (42,436)(1 + 0.03(1)) = (42,436)(1.03) = 43,709.08$$

Finally, at the end of year four Rich has

$$A = (43,709)(1 + 0.03(1)) = (43,709)(1.03) = 45,020.35$$

truncating to the penny. This isn't bad, especially if you consider that he has made \$ 5,020.36 by doing almost no work at all. However, that was a lot of math—isn't there an easier way?

Luckily, there is a short cut formula, which we will now learn about.

Let's try to examine the previous example from the standpoint of algebra. We can write A_1, A_2, A_3, and A_4 for the amount after the end of each year in the previous problem. The principal in Year 4 is A_3, in Year 3 the principal is A_2, in Year 2 the principal is A_1, but in Year 1 it is the original principal P. This paragraph was a bit wordy, so take a moment now to go back and reread it. We're going to use these four As momentarily.

We have four formulas, which we will combine in the next box. Note, $t = 1$ in each case, and we'll just leave r as a variable for now.

$$\begin{aligned} A_4 &= A_3(1 + r1) = A_3(1 + r) \\ A_3 &= A_2(1 + r1) = A_2(1 + r) \\ A_2 &= A_1(1 + r1) = A_1(1 + r) \\ A_1 &= P(1 + r1) = P(1 + r) \end{aligned}$$

First, let's combine

$$A_1 = P(1 + r) \text{ and } A_2 = A_1(1 + r)$$

which gives us

$$A_2 = P(1 + r)(1 + r)$$

We can combine that newest equation with

$$A_3 = A_2(1 + r)$$

to obtain

$$A_3 = P(1 + r)(1 + r)(1 + r)$$

That's progress. We can combine this latest equation with

$$A_4 = A_3(1 + r)$$

to obtain

$$A_4 = P(1 + r)(1 + r)(1 + r)(1 + r)$$

In the previous box, we had the formula

$$A_4 = P(1+r)(1+r)(1+r)(1+r)$$

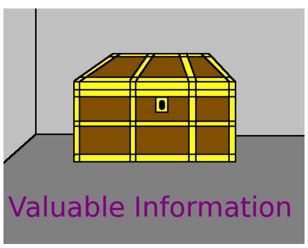
which allows us to solve our previous example for any interest rate r and any principal P, but for $t = 4$ years. First, we can simplify this formula. If you think how exponents work then you know we can write this as

$$A_4 = P(1+r)^4$$

which is a lot more compact.

Second, if we want to consider values of t other than $t = 4$, the change that we should make is obvious:

$$A_t = P(1+r)^t$$



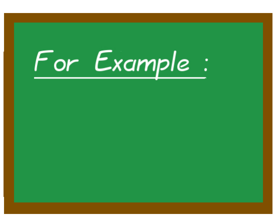
Now using the theory, we can understand the following formula for compound interest

$$A = P(1+i)^n$$

where P is the principal and A is the amount at the end. The meaning of n is the number of compounding periods. Usually loans are compounded monthly, but sometimes annually or weekly. There are occasionally loans that are compounded quarterly, biweekly, or semiannually. In our previous problem, we had year-long CDs, so that means the compounding is annual. The previous example had 4 years, then, we have $n = 4$ periods. If it were monthly, then 4 years would be $n = 4 \times 12 = 48$ periods.

You're probably surprised to see that I'm using i here for interest instead of r. The distinction is that r should be used *only for annual rates of interest*, but i is the interest rate per period, which we will explain by examples throughout this module. This will become clear, shortly.

The compound interest formula is one of the most important formulas *in the entire book*, so learn it well!

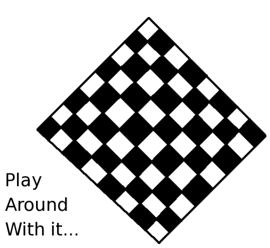


2-2-2

Returning to previous example of buying a car, let's consider the possibility that Rich is impatient, and wants to drive the car during his senior year of college. This means that there will only be three years worth of CDs, and not four years. The principal is still all of his inheritance or \$ 40,000. The interest rate we will take as 3% again. Using the formula $A = P(1+i)^n$ we plug in our data and get

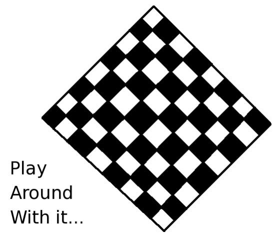
$$A = (40,000)(1.03)^3 = (40,000)(1.092727) = 43,709.08$$

Therefore, Rich will be limited to \$ 43,709.08, which isn't bad at all.



2-2-3

Now consider if Rich had received \$ 50,000 instead, but that he stays in college for 5 years. (Perhaps he's studying architecture, which is a five year degree, or perhaps Rich just drank too much while attempting a four-year degree.) The interest rate is still 3%. How much does Rich have to spend on his car at the end? [Answer: \$ 57,963.70]



2-2-4

Again, what if Rich received $ 50,000, but stays in college for 4 years. Suppose the interest rate is 5% on the certificates of deposit. How much does Rich have available to spend on his car, at the end of those four years? [Answer: $ 60,775.31.]

Many students, when they see $A = P(1+i)^n$ will want to replace the $(1+i)^n$ part with $1^n + i^n$. This is definitely not allowed! Just to be clear,

$$A = P(1+i)^n \neq P(1^n + i^n)$$

Consider just a simple test on your calculator. Perhaps we shall choose $P = 30,000$, $n = 10$ and $i = 0.06$. What happens in this case?

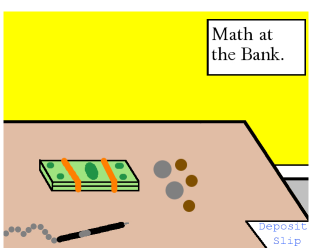
The $(1+i)^n$ version of the formula becomes

$$(30,000)(1+0.06)^{10} = (30,000)(1.79084\cdots) = 53,725.43$$

However, the $1^n + i^n$ version becomes

$$(30,000)(1^{10} + 0.06^{10}) = (30,000)(1.00000\cdots) = 30,000.00$$

Therefore, we can see that the act of replacing $(1+i)^n$ with $1^n + i^n$ is absolutely prohibited, because it will give you wrong answers!

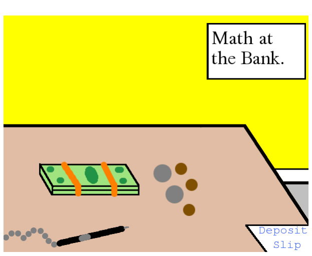


The periods for loans have a standard terminology, which is easy but important to memorize. For example, "quarterly" means that the loan or investment compounds four times per year. To represent this, we would say $m = 4$. Monthly, naturally, means that the loan compounds once per month, and we write this $m = 12$, because there are twelve months per year.

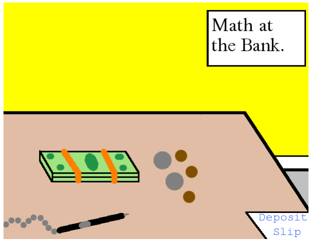
Period	m	Period	m
Annually	1	Monthly	12
Semiannually	2	Biweekly	26
Quarterly	4	Weekly	52
Bimonthly	6	Daily	360

We'll consider the case of daily below.

Observe that in all cases, m is the number of compoundings per year.

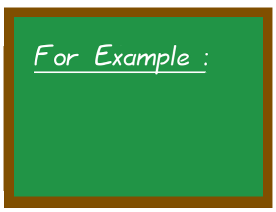


An interesting point is whether the year should be considered to be always 365 days, always 360 days, or 365 days for non-leap years and 366 days for leap years. The most common convention is to assume that all months have 30 days, which is obviously false, and then use 360 days for the year, which is false as well. That convention is called Banker's Rule, and we will use this as our rule in this book. While it is based upon two lies, on the other hand, it makes the calculations easier.



There are other standards, and an excellent discussion of this issue of the number of days in the year for financial problems can be found in the book *An Introduction to the Mathematics of Money—Saving and Investing* by David Lovelock, Marilou Mendel and A. Larry Wright, Chapter 1, Section 2.

The difference involved is a few pennies, therefore this matter is not worth further discussion. Admittedly, it would be very annoying to have adjust by 28, 30, or 31 depending on what month of the year it is.



2-2-5

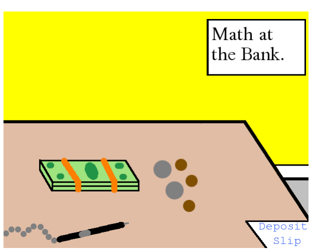
Most investments are not compounded annually, but instead monthly. Let's explore that now. Returning to the case of Rich saving up for a nice car, if his savings account compounds monthly, then how does the problem change?

Let's suppose that Rich still has his $ 40,000 inheritance, and the interest rate is still 3%. (When people say the interest rate is 3%, they mean per year.) If he saves for 4 years, then that's $4 \times 12 = 48$ months.

However, for i, we must not use $i = 0.03$, because 3% is the interest rate per year. Instead, we must use the *interest rate per period* or $0.03/12 = 0.0025$, because there are 12 months in the year, and the period on the account is one month. That means $i = 0.0025$, or 1/4th of 1% per month. Now with the formula $A = P(1+i)^n$ we obtain

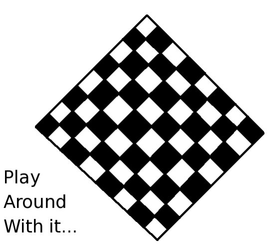
$$A = (40,000)(1+0.0025)^{48} = (40,000)(1.12733\cdots) = 45,093.12$$

By the way, the interest rate per period is sometimes called the *periodic rate*.



Interestingly, we see that changing the compounding from monthly to annually, but leaving everything else the same (3% interest, and a $ 40,000 principal), resulted in a small difference—namely $ 45,093.12 compared to 45,020.35 from the very first example box in this module. That difference comes to $ 72.77. Certainly large enough that a bank employee would be in trouble for being off by this much, but not large enough to affect financial planning.

The frequency of compounding is of secondary—but non-zero—importance.



2-2-6

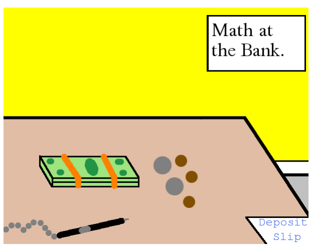
- Let's consider now the case that the interest rate is 4%. We still are compounding monthly, saving for 4 years, and depositing $ 20,000. What is the amount at the end? [Answer: $ 23,463.97].
- Let's consider now the case that the interest rate is 5%, but everything is the same as the previous bullet. [Answer: $ 24,417.91].

Interest rates are very sensitive numbers. A slight change in the interest rate can have catastrophic consequences to any calculation. To see this, let's take an example. (You saw this before on Page 45, in the module "Scientific Notation and Significant Figures," but now perhaps you can understand it better.)

Money is to be put away when a child is born, for his college education. Thus the account will run for 18 years, and suppose it has a 7% interest rate. Perhaps \$ 10,000 is deposited, and the account will compound monthly. The interest rate per month would be $0.07/12 = 0.00583\overline{3}\cdots$.

Now suppose three students are doing this, namely Albert, Beatrice and Christopher. Beatrice will dutifully use the exact value of i to as many decimal places as her calculator allows, but the boys round off. Albert will use 0.0058, and Christopher will use 0.006. What are the consequences?

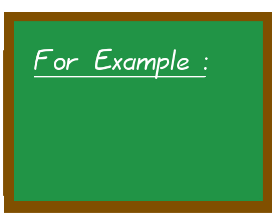
- For Albert, $A = 10,000(1 + 0.0058)^{18\times 12} = 10,000(1.0058)^{216}$ $= 10,000 \times 3.48749\cdots = \$\ 34,874.85$.
- For Beatrice, $A = 10,000(1 + 0.005833\overline{3})^{18\times 12} = 10,000(1.005833\overline{3})^{216}$ $= 10,000 \times 3.51254\cdots = \$\ 35,125.39$.
- For Christopher, $A = 10,000(1 + 0.006)^{18\times 12} = 10,000(1.006)^{216}$ $= 10,000 \times 3.64052\cdots = \$\ 36,405.23$.



Let's examine the answers from the previous box.

As you can see, the difference is enormous! Albert is off by \$ 250.54, but Christopher is off by \$ 1279.84. It takes at most 2 seconds, and in all probability much closer to 1 second, to write down all the digits that your calculator provides when working with an interest rate. Be certain *never* to round off interest rates.

Never round off an interest rate under any circumstances!



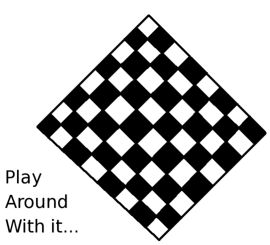
2-2-7

Chris's friends have recently had a child. He wants to help them with the child's education. He gives them a one-time gift of \$ 5,000. Assuming the parents invest it, compounding quarterly at 5% per year, how much will they have when the child turns 18 years old?

Clearly $r = 0.05$, and quarterly means $m = 4$. So then we have $i = 0.05/4 = 0.0125$. Next, we have $P = 5000$. There are $18 \times 4 = 72$ quarters in 18 years. Now we have

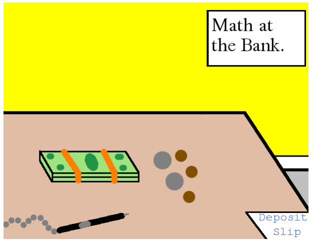
$$A = (5000)(1 + 0.0125)^{72} = 5000(1.0125)^{72} = 12,229.60\cdots$$

and we learn that the five grand will become almost triple the original value.

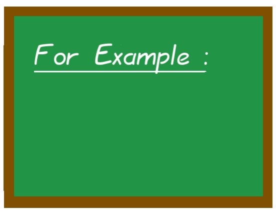


2-2-8

- Repeat the previous example with biweekly, instead of quarterly compounding. [Answer: $\$\ 12,287.39\cdots$].
- Repeat the previous example with semiannually, instead of biweekly, compounding. [Answer: $\$\ 12,162.68\cdots$].



As you can see, there is a huge change whenever i or r is modified slightly, but at the same time, there is very little change whenever m is modified slightly. Part of working with formulas is knowing how the answer depends on each variable. From this we learn that compound interest is very sensitive to interest rate, but not very sensitive to compounding frequency. This means when you are shopping around for a loan or an investment, you should be very selective about interest rates, but need not worry as much about the compounding schedule. This concept will be made precise on the module "Predatory Lending and the Annual Equivalent Rate," which begins on Page 404.



2-2-9

Suppose I see an ad for certificates of deposit, compounding quarterly and paying 3% interest. Imagine that I want to buy a boat for \$ 10,000 exactly 2 years from now, and I want to know how much I should deposit now, to make that possible.

Well, the \$ 10,000 is definitely A, as that is the amount at the end of the loan. Next, the compounding is quarterly, so $m = 4$, and we are given that $r = 0.03$. Furthermore, this means $i = 0.03/4 = 0.0075$. The loan will have 8 quarters (2 years of 4 quarters each), so that is $n = 8$. We now have the equation

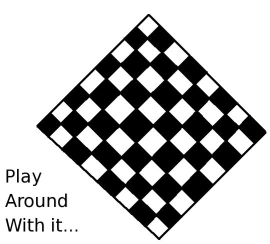
$$10,000 = P(1 + 0.0075)^8$$

which we will solve in the next box.

In the previous example we had the following equation, which we will now solve.

$$\begin{aligned} 10,000 &= P(1 + 0.0075)^8 \\ 10,000 &= P(1.0075)^8 \\ 10,000 &= P(1.06159\cdots) \\ \frac{10,000}{1.06159\cdots} &= P \\ 9419.75\cdots &= P \end{aligned}$$

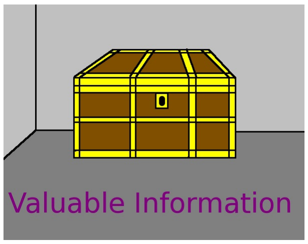
Therefore, I must deposit \$ 9419.75 now in order to buy that boat in two years.



2-2-10

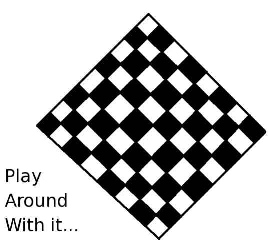
- Repeat the previous example, with $r = 4\%$.
  [Answer: \$ 9234.83 · · · .]
- Repeat the previous example, with $r = 4\%$ but 3 years instead of 2 years.
  [Answer: \$ 8874.49 · · · .]
- Repeat the previous example, with $r = 4\%$ but 4 years instead of 2 years.
  [Answer: \$ 8528.21 · · · .]

As you can see, compound interest is much more effective when it is given some time to work and grow. By waiting four years, I'd save $10,000 - 8528.21 = 1471.79$ dollars off the cost of the boat.



We've used a lot of variables in this section, but we'll be using the same ones over and over throughout this chapter, and the next chapter too. Actually, most finance books printed in the English language use roughly the same lettering system. The variables are

- The amount of the loan, or A. Remember, this is the quantity of money at *the end* of the loan. The memory hook is that you would like to receive an "A" at the end of this course.
- The compounding frequency m, which is the number of times per year that the loan or investment is compounded. Why is it m? Because $m = 12$, for compounding monthly, is a very common situation. Thus "m" for "monthly."
- The number of times that the loan will compound is n. Think "n" for "number."
- The principal of the loan, or P. Remember, this is the quantity of money at *the start* of the loan. The memory hook is "*princeps*" which means "first" in Latin. That's where the noble title "Prince" comes from, the first son of a King.
- Then there is the rate of interest, which can be written two ways. The interest per year is the *nominal rate*, *quoted rate* or *published rate*, and we write r for that.
- The interest rate per period is what you use inside the formulas, and that is $i = r/m$. It is sometimes called the *periodic rate.*
- The meaning of t is "time," but beware, it must always be measured in years.
- If the loan is t years long, then $n = mt$. Often the number of compoundings is obvious, so there is no need for a new variable t and a calculation to find n.



2-2-11

Suppose that a company is building a shopping mall, and that the construction project has gone over budget. They discover that they are 40 million dollars short of being able to complete the mall. This is a huge problem, because no shops will move into the mall and start paying rent until the mall is entirely completed!

The company shops around for what is called a *bridge loan*. Just a bridge gets you from one side of a river to another—a trip that isn't very long, but that is impossible with out a bridge—a bridge loan allows a business to cross some unpleasant financial obstacle for a relatively short amount of time.

In this case, suppose that the company decides that 36 months should be plenty, and they get an offer for 12% compounded monthly. (That's a bit high, but bridge loans often have a high interest rate, since the companies are desperate.)

After the 36 months are over, the mall will be long since completed, and presumably lots of shops will have moved in and will be paying rent. What payment will the company have to make at that time?
[Answer: $ 57,230,751.34.]

Amazingly, one time when I gave the previous question to a large Finite & Financial Mathematics class at UW Stout, I saw about a fifth of the students give me an answer of

$$\$\ 2,943,699,441.28$$

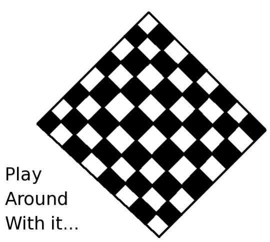
which is obviously completely absurd. At just under 3 billion dollars, that's almost the cost of a nuclear-powered aircraft carrier, not a shopping mall! Who would accept the terms of such a loan?! Even if someone were to be so foolish as to accept such terms, where would they get the 3 billion dollars for paying it back?!

In the workplace, such a mistake could demolish your credibility forever. After such a mistake, it is reasonable to imagine that a junior-level employee would simply not be renewed at the company, and would be let go at the expiration of their contract. I'm sure we can all agree that \$ 57 million dollars and \$ $2.943\cdots$ billion dollars are not synonyms!

After I calmed down a bit, I tried to figure out what could have caused the mistake which I described in the previous box. Some students will see "36 months" and use $n = 36$. Other students will realize that 36 months is the same as 3 years, and will write $t = 3$ and $m = 12$, to obtain

$$n = mt = 36$$

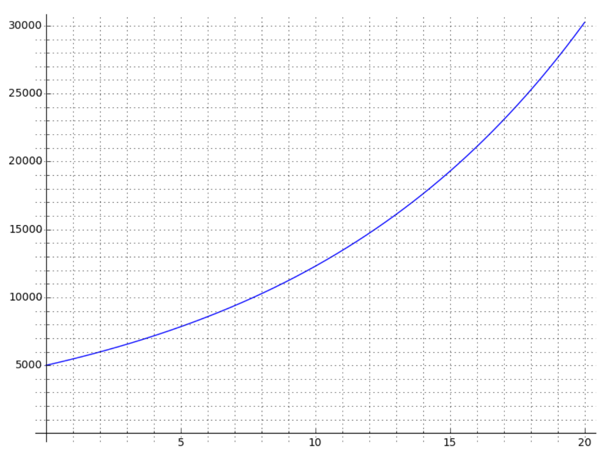
which is also a correct approach. However, a few students wrote $t = 36$, which does not signify 36 months—it signifies 36 years, and that's very, very different.



2-2-12

Let us suppose that I have a debt of \$ 5000 which compounds daily, at the rate of 9%. Remembering to use a 360-day year, compute what I would have to pay back, for the following time intervals.

- After 5 years? [Answer: \$ 7841.11.]
- After 10 years? [Answer: \$ 12,296.63.]
- After 20 years? [Answer: \$ 30,241.43.]



The graph to the left represents the growth of funds from the previous box. Specifically, I've graphed

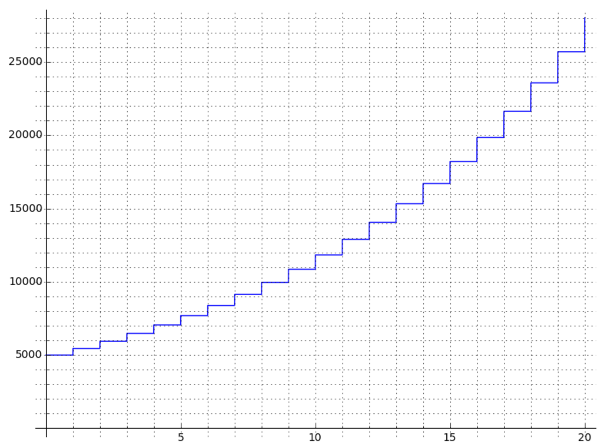
$$A(t) = 5000\left(1 + \frac{0.09}{360}\right)^{360t}$$

for the years $0 < t < 20$, representing a debt of \$ 5000 at 9% compounded daily.

Just to be clear, the function $A(t)$ is taking the role of y and the variable t is taking the role of x.

Earlier, we saw on Page 268 that the graph of simple interest over time is just a line, but that's clearly not the case here. Because the graph is curved, you might imagine it to be a parabola, but it is actually something called an "exponential curve." Don't worry about that terminology for now, but try to remember that shape.

Also, notice that it is much steeper at the end than at the beginning, which you can tell by looking at the slope.

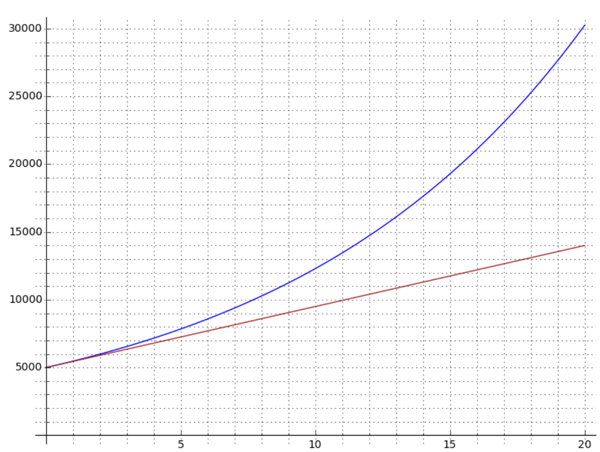


The graph to the left represents the growth of \$ 5000 at 9%, compounded annually. You might be wondering why it looks like a staircase.

The reason it looks like a staircase can be explained by considering $t = 3.1$, $t = 3.3$, or $t = 3.7$ just as examples. In each of these cases, the investment has grown from the 1st, 2nd, and 3rd compounding, but it has not grown from the 4th compounding. The fourth compounding must occur only when $t = 4$, not even at $t = 3.998$. Therefore, for all values of t with $3 \leq t < 4$, it is as if $t = 3$.

That's why, for $3 \leq t < 4$, the graph looks like a horizontal line. The same is true for the other intervals. Therefore, the graph looks like a collection of 20 horizontal line segments, representing the 20 years in the problem. Together, the 20 line segments look like a staircase! Last but not least, did you notice how the steps toward the right of the graph are much steeper than the steps toward the left of the graph?

By the way, you'd be amazed at how many Finite Mathematics textbooks get the previous graph incorrect. I've seen it in many well-known textbooks, where they will use a smooth curve to represent annual compounding. (In other words, they will use a graph that looks like what I drew for "daily" even for a situation like "annually.") I would like to thank my colleague Prof. Seth Dutter, for encouraging me to include this important point.



The graph at the left compares 9% compounded daily with 9% simple interest, again with a \$ 5000 principal.

As you can see, near the beginning, there isn't all that much difference between simple interest and compound interest. They are growing at about the same rate, which you can see because the graphs are so close together. Yet, by the end of 20 years, the difference is huge. Compound interest can be very powerful, but it also needs time to work.

Once again, we can see that the wealth is growing faster toward the end than at the beginning. That's because the slope of the line and the exponential curve are about the same at the beginning, yet at the end they have very different slope. We will have more to say about graphs of this type on Page 428.

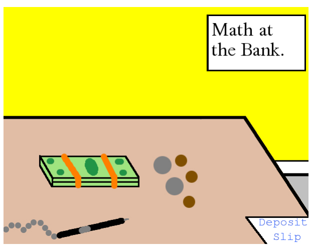
Finally, note that in all three graphs, the graph at $t = 0$ is showing the principal. If you plug $t = 0$ into either the simple interest formula or the compound interest formula, you get $A = P$. For example, in simple interest we have

$$A = P(1 + rt) = P(1 + r0) = P(1 + 0) = P(1) = P$$

while in compound interest we have

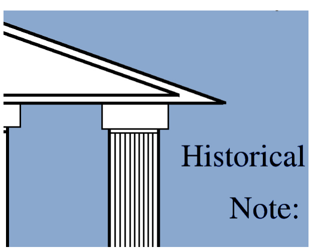
$$A = P(1 + i)^n = P\left(1 + \frac{r}{m}\right)^{mt} = P\left(1 + \frac{r}{m}\right)^{m0} = P\left(1 + \frac{r}{m}\right)^{0} = P(1) = P$$

In these graphs, $P = 5000$, so the y-intercept is always \$ 5000.



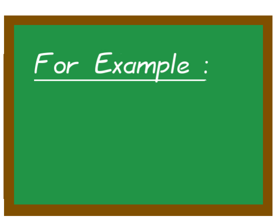
While interest rates are almost always positive, there are bank accounts that pay a negative interest rate. These occur in nations known for "private banking," such as Switzerland, the Cayman Islands, or St. Nevis & St. Kitts. The idea is that if ordinary savings accounts are paying +0.50%, then perhaps certain people would be willing to cheerfully accept −1.50% for an account with no name—just an account number.

This means that the tax offices will know nothing about this account, and that the wealth in it can be kept very private. As you can see, this is ideal for tax evasion, organized crime, and money laundering activities.



These accounts are so private that military dictators love them. One notable example was Fulgencio Batista (1901—1973) who was the dictator of Cuba from (1952–1959). In 1959, he fled to Portugal with large sums of money and lived out the rest of his life in luxury, writing books. Meanwhile another dictator, Fidel Castro (1926–present), took over Cuba.

In the years when Batista was a dictator, he could do what he wanted, including manipulate prices. He once made a somewhat large increase in the telephone rates charged in Cuba. The American phone company involved was so grateful for this that they gave him—as a gift—a telephone made out of gold.



\# 2-2-13

Suppose a criminal has secretly deposited five million dollars in a Swiss bank account, that has a rate of −1.50%. Unfortunately, he is caught and sentenced to 7 years in prison. How much is in the account when he is released? For simplicity, let us assume that this is compounded annually. He would have

$$\begin{aligned} A &= P(1+i)^n \\ &= (5,000,000)(1-0.015)^7 \\ &= (5,000,000)(0.985)^7 \\ &= (5,000,000)(0.899608\cdots) \\ &= 4,498,043.15 \end{aligned}$$

which is not a small sum of money. As you can see, the effect is a slow evaporation of money, and we will see a similar effect when we learn about radioactive decay on Page 474, where radioactive chemicals decay slowly into other chemicals.

Play Around With it...

Continuing with the previous box, suppose his accomplice used a bank in the Cayman Islands which had a rate of -1.25%, compounded monthly. Luckily, he only got a 5-year prison sentence. If he deposited 2.5 million dollars, how much is waiting for him when he is released? [Answer: $ 2,348,456.15.]

2-2-14

In the next box, a young man named Neil is about to propose marriage to his long-time sweetheart. He goes to a jewelry shop to get a suitable ring, and the salesperson tells Neil that he should spend an amount equal to two months salary on the ring. He is just starting out in life, being a recent college graduate, so he makes $ 48,000 per year—that comes to $ 4000 per month. Isn't $ 8000 a lot of money for a bit of jewelry?

Neil goes home to think about this, and decides upon a wiser strategy, which we will explore in the next box.

For Example :

Neil will buy a very simple featureless silver hoop for $ 5 instead, and deposit the remaining money ($ 7995) in a mutual fund. He will spend that money when he retires at age 65. He is currently 23 years old. How much money will be in the mutual fund at that time, assuming that the yield on the mutual fund is 9% per year? We will use annual compounding on this problem.

The principal is $ 7995. Next, since Neil is 23, then he will be 65 in $65 - 23 = 42$ years. And the interest is given as 9% per year, and we are compounding yearly, so $i = 0.09$. Now we plug in to obtain

$$A = (7995)(1 + 0.09)^{42} = 298,353.66$$

Therefore, we learn that the young man will have, when he is ready to retire, well over a quarter of a million dollars in return for not wasting money on an engagement ring. He can easily take his wife on a trip around the world, and have plenty left over for many other luxuries.

2-2-15

For Example :

A common thought experiment in books teaching financial mathematics is to consider what would have happened if an ancestor of yours deposited $ 100 in a bank account during the American Revolution. Let us consider that the deposit may have been made in July of 1776, and let us ask its value in February of 2011. In that 235 years and 6 months, surely the interest rate would have fluctuated greatly. For simplicity, let us select 3% compounded monthly. What would the account have in it today?

$$A = P(1+i)^n = 100\left(1 + \frac{0.03}{12}\right)^{235\times12+6} = 100(1.0025)^{2826} = 116,000.94$$

which is a very different number than $ 100.

2-2-16

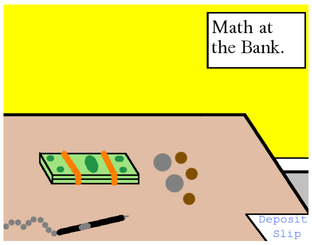
Play Around With it...

It is much more exciting to imagine the money having been invested in the stock market, where the rate of return would be much higher. Of course, the rate of return of the stock market, regardless of what index you look at, is anything but constant. Stock market indices are remarkably, even famously, unpredictable. While buying stocks did exist as a form of investing in 1776, the activity did not become common until the 1880s. What would the final amount be if 7% had been the rate of return over that 235 year and 6 month period? [Answer: $ $1.37576\cdots$ billion dollars.]

2-2-17

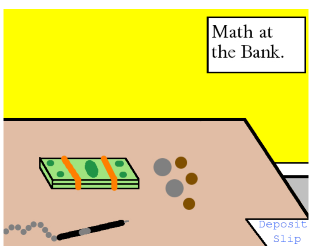
We are overlooking an important concept in the previous two boxes. That concept is inflation. You could buy a lot of things for $ 10 back in 1776, and you can buy far less for $ 10 now. We will explore this topic in tremendous detail beginning on Page 455.

Nonetheless, even after adjusting for inflation, these numbers would still be huge. It is because of this enormous power of compound interest over long periods of time, that when Albert Einstein (1879–1955) was asked what force in the universe was the most powerful, he replied "compound interest."



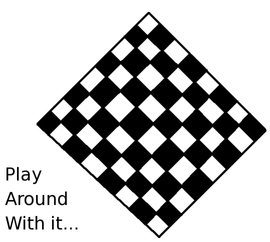
Now let's discuss savings bonds. These are an extremely safe and predictable way of saving money, without risk. On the other hand, the interest rate provided is often very meager compared to other investments, including those that carry fairly low risk.

Savings bonds should not be confused with treasury bonds. Treasury bonds are sold by the US Treasury because of the national deficit, and most are bought by large banks and pension funds. While savings bonds are also sold by the US Treasury, they are bought by individual citizens. The savings bond program exists to help citizens who cannot afford (or who do not desire) risky investments. Both kinds of bonds work semi-annually. Savings bonds use compound interest, whereas treasury bonds use simple interest.



The most common kind of savings bond issued by the US government is the EE-series bond, which lasts 30 years. The "face value" is a rough estimate of what the bond will be worth part-way through the bond's life. The principal of the bond (i.e. the price you pay for it) is the face value, divided by two.

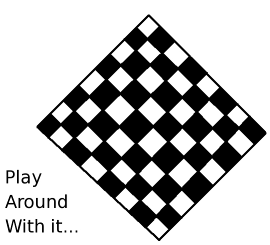
Some workplaces allow you to buy savings bonds directly from your paycheck. You select to withhold either a fixed amount on each payday, or alternatively, a percentage of your pay. Then bonds as small as $ 50 or as large as $ 2000 are purchased for you automatically, using that withheld money electronically.



2-2-18

Suppose a $ 20,000 face-value bond is bought at the start of a child's life, to help be a down-payment on his house when he is 30 years old. Suppose further that the interest rate is 4.5%. Like all federal bonds, this bond is compounded semi-annually. How much does the child have on his 30th birthday? [Answer: $ 38,001.35.]

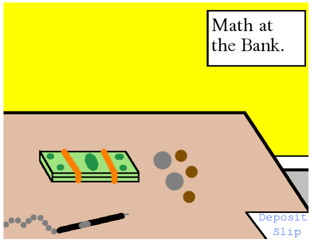
Note, that the face value (printed on the bond) is double the value of the principal. Since our bond had a face value of $ 20,000, that means that the principal (what you pay to buy it) is $ 10,000. This is a weird convention, but that is the convention.



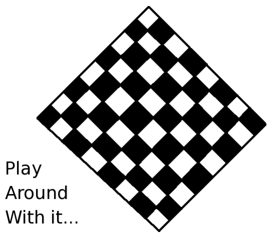
2-2-19

Repeat the above example with interest rates 5% and 4%, all other factors being the same. [Answer: $ 43,997.89 and $ 32,810.30.]

As you can see, savings bonds are not all that bad of an investment. A wealthy relative contributes $ 10,000, but when the baby grows up and turns 30, he will receive between $ 32,810.30 and $ 43,997.89, depending on the interest rate.



The next five boxes represent a true story. It was explained to me when I discovered myself upgraded to business class unexpectedly on a flight in January of 2011, and I found myself seated next to a high-ranking financier in the CFO's office of a major conglomerate. Naturally, we discussed this book that I am writing. He suggested this problem. It highlights why businesses merge to form conglomerates, and how the several businesses of a conglomerate can help each other.

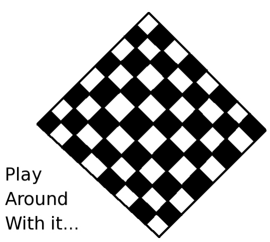


2-2-20

The problem begins with two businesses owned by the conglomerate (these are called subsidiaries). I regret that I cannot use the real names.

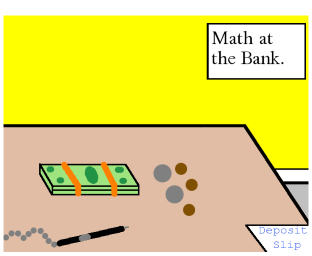
- Company A has some funds (forty million dollars) for the construction of a shopping mall, but due to the climate there, work cannot begin until March 31. Accordingly, in late September, Company A gets a 180-day Certificate of Deposit, and their money earns 1.5%, compounded monthly. (The interest rates were very low at that time.) How much interest do they earn by this certificate, instead of leaving the money as cash? [Answer: $ 300,939.06.]
- Company B requires additional funding for a project, also 40 million dollars and for six months, again in late September. Because this is also a construction project, and they can put the land down as collateral, they are able to get a good rate, of 3% compounded monthly. How much interest do they pay at the end of the loan? [Answer: $ 603,762.52.]

During audits of the books, the finance office of the conglomerate notices these two activities of its subsidiaries. Company A's certificate of deposit has no early withdrawal penalty, therefore Company A withdraws and loans the 40 million dollars to Company B, which no longer needs the bank loan as a result. They agree on a rate of 2.25%, compounded monthly. Getting the timing correct would make this problem confusing, so let's imagine that headquarters had noticed the opportunity instantly, before the certificate of deposit had been made, and the loan contracted for.



2-2-21

Continuing with the last few boxes, Company A loans 40 million dollars to Company B for 180 days, at a rate of 2.25%, compounded monthly. How much does Company B pay in interest for this loan? [Answer: $ 452,114.65.]

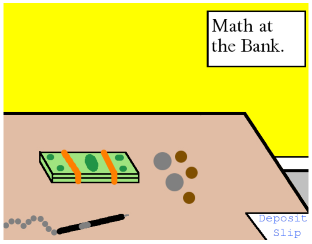


Now we're going to look at what happened from the perspectives of each company, and the conglomerate. The first point of view is Company A.

Company A has 40 million dollars that it will not be using for 180 days. They shop around for certificates of deposit, and find one opportunity that will earn them $ 300,939.06. Then their corporate headquarters notifies them of an opportunity, where they can loan the money to Company B, and earn $ 452,114.65 instead. Under this arrangement, Company A ends up with

$$452,114.65 - 300,939.06 = 151,175.59$$

more than planned, therefore Company A is very happy.

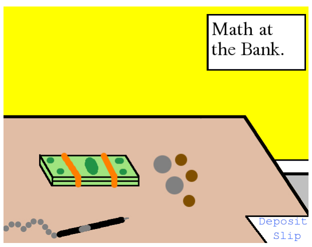


The second point of view is Company B.

Company B has a shortfall of 40 million dollars in its budget for a project, and needs the money for 180 days. They shop around for bridge loans, and the best offer they get from a bank will have them paying \$ 603,762.52 in interest, which isn't bad. However, they get a "suggestion" from their corporate headquarters of an inter-subsidiary loan, which would have them paying \$ 452,114.65 instead. Under this arrangement, Company B saves

$$603,762.52 - 452,114.65 = 151,647.87$$

therefore Company B is very happy.

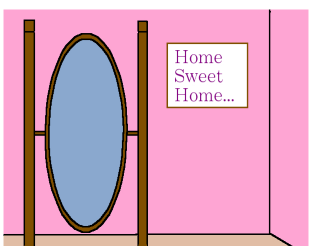


The third point of view is of the conglomerate itself.

Prior to setting up the inter-subsidiary loan, the conglomerate sees Company A gaining \$ 300,939.06 from the banking community, but Company B paying \$ 603,762.52 to the banking community. Therefore a net loss of

$$603,762.52 - 300,939.06 = 302,823.46$$

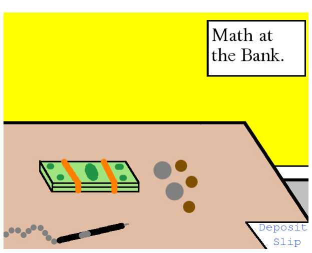
of the conglomerate's money is going from the conglomerate into the banking community. After making the suggestion, the subsidiaries just move money amongst themselves, which is irrelevant from the conglomerate's point of view. Thus, the conglomerate is very happy—having saved \$ 302,823.46.



A Pause for Reflection...

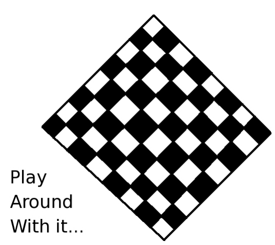
Surely we can all agree that it is a very rare circumstance that there would be three parties to a business transaction, and all three are very happy with the outcome. Of course, the bank which would have sold Company A a certificate of deposit, and the bank which would have given Company B a bridge loan, are both disappointed.

While the previous few boxes highlighted a complex situation, it is a realistic situation. Perhaps you'd like to take a few moments to go back and read through all that again.



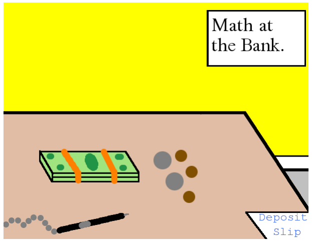
One of the first US pennies ever made was sold by a coin collector in an auction on January 7th, 2012. The coin was struck with the date 1793. Can you guess the sales price?

It sold for \$ 1,380,000. This sounds like a phenomenal rate of growth, for something originally intended to be worth \$ 0.01. Intuition says that it is a phenomenal rate of growth, in any case. We're going to explore this situation in detail now, over the next three boxes.



2-2-22

- If someone invested \$ 0.01 in 1793, at 8% compounded quarterly, and a descendant of theirs withdrew the money in 2011, how much would they have? [Answer: $\$ 315,754.\cdots$.]
- What if it were at 9% compounded quarterly? [Answer: $\$ 2,669,390.\cdots$.]

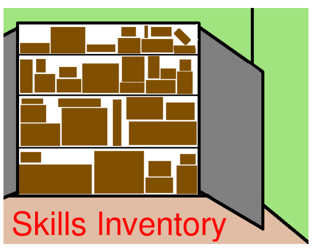


Looking at the previous box, you can see that invested at 8% compounded quarterly, the penny would have made less money, and at 9% compounded quarterly, the penny would have made more money. Therefore, the equivalent rate of return must be somewhere between 8% and 9%. On Page 351 we'll see how to calculate exactly what the rate of return was.

Many ordinary investments can have rates of return in the range of 8%–9%, so the value of that coin really hasn't experienced a *phenomenal* rate of growth, just a good rate. This is yet another example of how intuition can mislead, and how we must rely on calculation.

Here is one last minor note about the penny problem. While analyzing the penny problem over the last three boxes, why did we use 2011 and not 2012? Well, the penny was struck in 1793, so that could be done anywhere from January 1 to December 31 of that year. Likewise, January 7th (the date of the auction) is really early in 2012. So it seems fair to "shave off one year." This is only an estimate, after all.

However, this is not a point which should concern us much at this time.



We have learned the following skills in this module:

- To solve the compound interest formula $A = P(1+i)^n$, when A is unknown.
- To solve the compound interest formula $A = P(1+i)^n$, when P is unknown.
- To handle the many types of compounding: annually, semiannually, quarterly, bimonthly, monthly, biweekly, weekly, and daily, by the use of the variable m.
- To perform calculations involving EE-series US Savings Bonds.
- As well as the vocabulary terms: bridge loan; CD; certificate of deposit; checking account; compound interest; nominal rate; periodic rate; published rate; quoted rate; savings account.