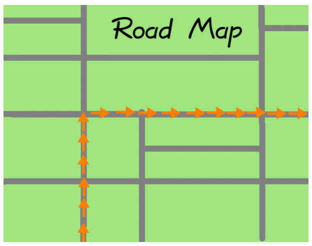
Module 2.7: How Exponents Really Work

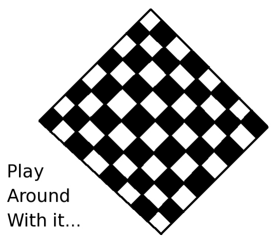


While you already know that $a^2 = a \times a$ and $a^3 = a \times a \times a$, in this section we're going to explore what it means for an *exponent* to be a fraction, or negative, or zero. This will become important in solving certain types of problems relating to loans.

If you've seen this material before, I encourage you to read this module anyway, as it may highlight some links to economics and finance that would not have been included in previous math courses.

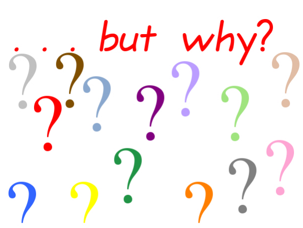
The laws of exponents are very useful. We're going to give you a list of these laws at the end of this module. You could memorize them if you really wanted to, I suppose. However, I find it is far easier to simply read the derivation from which these laws were born, and then if you take the small amount of time to actually think about them intellectually, you will discover that your brain retains each of the laws automatically after that. Or, you are welcome to just memorize them if you like.

Either way, you should take the time to practice with them. Every book numbers these laws differently, so there is no need to worry about which is the sixth law and which is the fourth law; just make sure you can remember and use them correctly.



2-7-1

If you aren't familiar with what a^4 or a^3 or even a^2 mean, take your favorite number, perhaps 42, and calculate $42 \times 42 \times 42$ on your calculator. Then calculate 42^3, and see that they are both 74,088.



There is a formula that says

$$x^a x^b = x^{a+b}$$

Let's see why that's true. Consider now x^2x^3; we can think of this as follows:

$$(x^2)(x^3) = (xx)(xxx) = (\underbrace{xxxxx}_{\text{5 times}}) = x^5$$

and likewise

$$(x^4)(x^5) = (xxxx)(xxxxx) = (\underbrace{xxxxxxxxx}_{\text{9 times}}) = x^9$$

which generalizes to

$$(x^a)(x^b) = (\underbrace{xx\cdots x}_{\text{a times}})(\underbrace{xx\cdots x}_{\text{b times}}) = (\underbrace{xx\cdots x}_{\text{a+b times}}) = x^{a+b}$$

and that is the first law of exponents.

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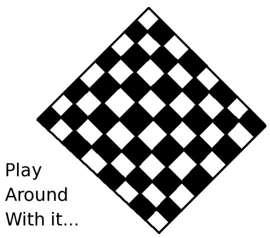
Note: we have just proven an important rule, but one that students often get wrong. Note that the following two things, while they look to the eye as very similar, are mathematically completely different:

$$x^{a+b} = (x^a)(x^b) \quad \leftarrow \text{TRUE !}$$
$$x^{a+b} = x^a + x^b \quad \leftarrow \text{FALSE !}$$

Consider $x = 5$, $a = 2$, and $b = 3$.

- If you do $x^{a+b} = (x^a)(x^b)$ then you get $5^5 = 5^2 5^3$ or $3125 = (25)(125)$, which is right.
- If you do $x^{a+b} = x^a + x^b$ then you would get $5^5 = 5^2 + 5^3$ or $3125 = 25 + 125$ which would mean $3125 = 150$, which would be wrong!



2-7-2

Let's consider the specific case when $x = 11$.

- What is 11^5? [Answer: 161,051.]
- What is 11^3? [Answer: 1331.]
- What is 11^2? [Answer: 121.]
- Is it true that $1331 \times 121 = 161,051$? [Answer: Yes.]
- So can we conclude that $(11^3)(11^2) = 11^5$? [Answer: Yes, we can!]

So now we ask ourselves, what should 11^0 be? No one can deny that $0 + 3 = 3$, and so $(11^0)(11^3) = (11^3)$. Now we don't quite know what 11^0 should be yet, so we'll call it y, but we know that $11^3 = 1331$ from the previous box. Thus we have $y1331 = 1331$, and we can then divide both sides by 1331, and get $y = 1$. Thus $11^0 = 1$.

Yet this isn't a property of 11 alone. It is a general property. Because $0+3 = 3$ we know that $x^0 x^3 = x^3$ must become $x^0 = 1$, because we just divide both sides by x^3, whatever that might be.

However, we are obligated to exclude $x = 0$, because that would be like dividing by 0^3, and you must never divide by zero! It would be mathematically meaningless to do so. Therefore, we do not know at this moment what 0^0 means, but we will explore that momentarily, on Page 355.

Thus, we can write $x^0 = 1$ for all positive numbers x, and this is the second law of exponents.

If you don't believe me, I suppose you could ask your calculator if $42^0 = 1$ or not, but rest assured 42^0 has equalled 1 from long before the invention of calculators!

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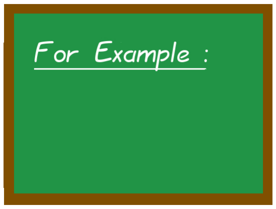
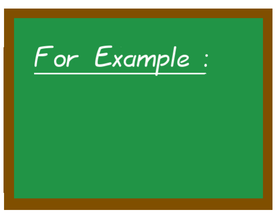


2-7-3

Returning again to the sports car from Page 276, if I get a \$ 20,000 inheritance, and want to save up for a sports car, then I need to decide how long I'm going to save up. At that point in the chapter, we had considered saving for 3 years or 4 years. Suppose I decide to invest it not for 3 years, nor 4 years, but for 0 years, then surely I would have only the \$ 20,000 and nothing more. Incidentally, the certificate of deposit returns 3% compounded monthly. Let's see what the formula tells us

$$A = P(1+i)^n = 20,000(1+0.03/12)^0 = 20,000(1.0025)^0 = 20,000(1) = 20,000$$

Of course, we didn't need any formulas to figure that out, but this example is a good memory hook for understanding what the 0th power really means.



2-7-4

We can verify with our calculator that $2^3 \times 3^3 = 6^3$, because that comes to $8 \times 27 = 216$, but what is really going on here? The following will make it clear.

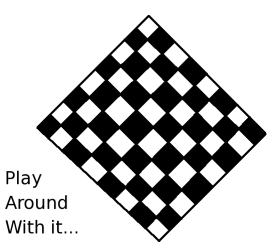
$$\begin{aligned} 2^3 \times 3^3 &= (2\times 2\times 2)\times(3\times 3\times 3) \\ &= 2\times 2\times 2\times 3\times 3\times 3 \\ &= 2\times 3\times 2\times 3\times 2\times 3 \\ &= (2\times 3)\times(2\times 3)\times(2\times 3) \\ &= 6\times 6\times 6 = 6^3 \end{aligned}$$

Of course, this is not a property just of 2, 3, and 6. Consider

$$\begin{aligned} a^4 \times b^4 &= (a\times a\times a\times a)\times(b\times b\times b\times b) \\ &= a\times a\times a\times a\times b\times b\times b\times b \\ &= a\times b\times a\times b\times a\times b\times a\times b \\ &= (a\times b)\times(a\times b)\times(a\times b)\times(a\times b) \\ &= (ab)\times(ab)\times(ab)\times(ab) = (ab)^4 \end{aligned}$$

This enables us finally to write the third law of exponents:

$$(a^x)(b^x) = (ab)^x$$



2-7-5

- What is 4^3? [Answer: 64.]
- What is 3^3? [Answer: 27.]
- What is $(3\times 4)^3$? [Answer: $12^3 = 1728$.]
- Is it the case that $64 \times 27 = 1728$? [Answer: Yes.]
- Can we conclude that $(4^3)(3^3) = 12^3$? [Answer: Yes.]

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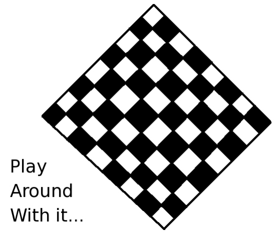
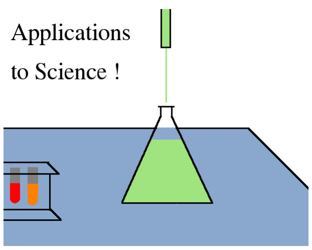


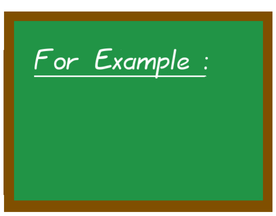
2-7-6

- What is 2^4? [Answer: 16.]
- What is 5^4? [Answer: 625.]
- What is $(2 \times 5)^4$? [Answer: $10^4 = 10,000$.]
- Is it the case that $16 \times 625 = 10,000$? [Answer: Yes.]
- Can we conclude that $(2^4)(5^4) = 10^4$? [Answer: Yes.]

We have now tested two specific cases of $(a^x)(b^x) = (ab)^x$. This is, of course, not a proof that it works for all numbers a, b, and x. However, you are welcome to test as many other specific cases as you like.



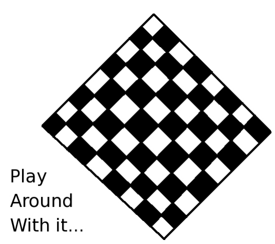
As it turns out, if you take an object, and make a scale model of it that is s times bigger in every way (height, width, and depth), then both the weight and the volume will be s^3 times as much. For example, if you triple all the measurements, then the weight and volume would be $3^3 = 27$ times as much. If you doubled all the measurements, then the weight and volume would be $2^3 = 8$ times as much. If you quadrupled all the measurements, then the weight and volume would be $4^3 = 64$ times as much.



2-7-7

A sculpture salesman has a very popular sculpture that is selling quite well: It is a 2-foot model of Michelangelo's David. He decides to try to sell one that is 3 times as big for 3 times as much money, because he thinks some large institutions like art schools might want a life-size 6-foot tall statue. However, shipping almost always has a cost proportional to the weight. The original statue has weight, perhaps, 100 pounds (45.3592 kg). How much will the life-size one weigh?

Well, it is 3 times bigger, so it will weigh $3^3 = 27$ times as much. That comes to 2700 pounds or 1224.70 kg—more than a ton! And this why you do not see life-sized statues very often at all. Moving something that heavy is *very* expensive.



2-7-8

Before making a sculpture, it is common for sculptors to make a scale model which is 1/4 as large. If the scale model weighs 50 pounds, how much will the final object weight? [Answer = 3200 pounds].

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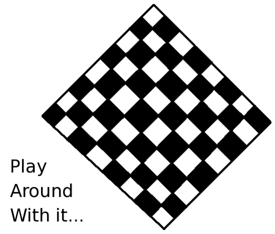
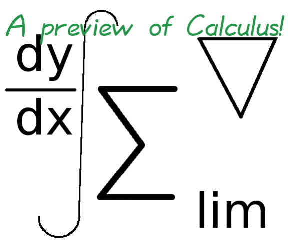


2-7-9

On the streets of Paris it is common to see peddlers selling tiny models of the Eiffel Tower to tourists. Suppose the standard one is 8 inches tall (20.32 cm) and weighs 2 pounds (907.185 grams). There is a smaller one that is only 4 inches tall. How much does it weigh? [Answer: 1/4 pounds or 113.398 grams].



In case you are curious, the reason for this trick with weights and volumes is the following: For a cube with side length s, the volume is s^3. In fact, that's why we would read aloud s^3 with the sounds "s cubed."

In calculus, one of the things that you can do is take a figure, and divide it into infinitesimally tiny cubes, and by adding up the volumes of those tiny cubes, you get the volume of the figure, via a long calculation. Image if you were to multiply the length of each of the sides of all the cubes by some number, like 4 for example. Essentially, you'd be changing the side length of each cube from s into $4s$. Then the volume of each cube would grow from s^3 to be $(4s)^3 = (4)^3 s^3 = 64s^3$. (Notice, we just used the third law of exponents.) If we had n of these tiny cubes, the volume would change from ns^3 to $64ns^3$. As you can see, the volume has grown to be 64 times as large.

Therefore, since each of the new cubes is exactly 64 times bigger/heavier than each of the old cubes, then the whole object is exactly 64 times bigger/heavier than the original.

Okay, how about negative exponents? Do we need them? It will turn out that you will probably never have occasion to raise a number to the 0th power during a financial calculation. Negative exponents, however, will occur often.

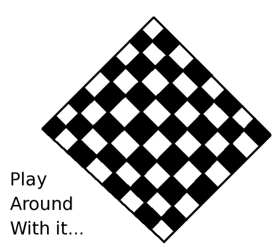
Consider then $(2^{-3})(2^3)$. By the first law exponents, we know $(2^{-3})(2^3) = 2^{-3+3} = 2^0 = 1$, the last step being due to the second law.

So $(2^{-3})(2^3) = 1$ and we know $2^3 = 8$, thus we have $(2^{-3})(8) = 1$. However, what multiplied by 8 will give me 1? Well, 1/8th of course! Therefore $2^{-3} = 1/8$.

This property is not unique to 2, but will work for any positive number. Let x be some positive number. We have

$$x^{-a}x^a = x^{-a+a} = x^0 = 1$$

and so therefore $x^{-a}x^a = 1$. Divide both sides by x^a (which is safe because we said x has to be positive, so it cannot be zero) and then we get $x^{-a} = 1/(x^a)$. This is the fourth law of exponents. Note that a consequence of this fourth law is that $x^{-1} = 1/x$.



2-7-10

The following should explain what negative exponents are really about.

- Without a calculator, what is 4^2? [Answer: 16.]
- With a calculator, what is 1/16? [Answer: 0.0625.]
- With a calculator, what is 4^{-2}? [Answer: 0.0625.]
- Can we conclude $\frac{1}{4^2} = 4^{-2}$? [Answer: Yes.]

As in the previous checkerboard, we've confirmed $x^{-a} = 1/(x^a)$ only for the case $x = 4$. However, you can try using any non-zero number for x, on your own.

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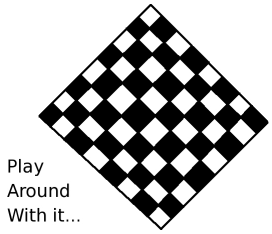
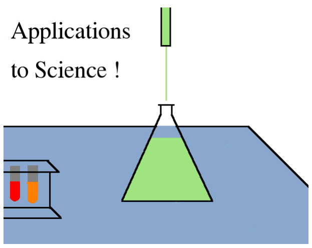


2-7-11

Here are two interesting thought-based questions:

- How do I write 1/9th as three raised to an exponent?
- How about 1/25th?

[Answer: 3^{-2} and 5^{-2}.]



The co-inventor of Calculus, Sir Isaac Newton, made numerous important discoveries. One of them was a theory of gravity. In fact, Newton discovered that the strength of gravity varies with distance according to the d^{-2} power. That is to say, if one person is twice as far from a planet as another, he or she feels $2^{-2} = 1/4$th as much pull. Likewise, if someone is 3 times as far from a planet as another, then he or she feels $3^{-2} = 1/9$th as much pull. The distance is always measured from the center of the planet. I suppose this should have been called the "negative two-th law", but it is called the "inverse square law" because that sounds better.

Several other physical forces obey the inverse square law, including electric charge, the intensity of light, and in certain circumstances, the volume of noise.

We will explore these matters (the inverse square laws) when we study "Non-Linear Proportions," the next module.

There are two different ways to understand what a^6/a^4 means. The first way is to regroup:

$$\frac{a^6}{a^4} = \frac{aaaaaa}{aaaa} = \frac{(\cancel{aaaa})(aa)}{\cancel{aaaa}} = \frac{aa}{1} = a^2$$

but another way is to remember that $1/a^4$ is the same as a^{-4}, and therefore

$$\frac{a^6}{a^4} = a^6\left(\frac{1}{a^4}\right) = a^6a^{-4} = a^{6+(-4)} = a^2$$

but naturally, we get the same answer either way. The third way is to look at it philosophically. As addition and subtraction are opposites, and multiplication and division are opposites, and since the first law says that multiplying a^6 and a^4 would lead to $a^{6+4} = a^{10}$ then likewise dividing a^6 by a^4 should lead to $a^{6-4} = a^2$. In any case, we are now certain that

$$\frac{a^x}{a^y} = a^{x-y}$$

and this is the fifth law of exponents.

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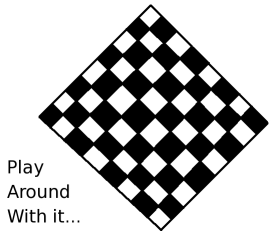
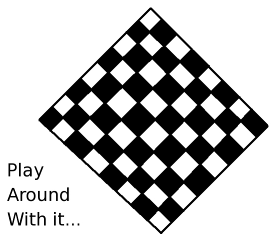


2-7-12

- What is 3^6? [Answer: 729.]
- What is 3^4? [Answer: 81.]
- What is 729/81? [Answer: 9.]
- What is $3^{6-4} = 3^2$? [Answer: 9.]

As you can see, we've checked the specific case of $a^x/a^y = a^{x-y}$ when $x = 6$, $y = 4$, and $a = 3$. We have not proven the general case, naturally, but that's more suited to a pure math course.



2-7-13

- What is 5^5? [Answer: 3125.]
- What is 5^3? [Answer: 125.]
- What is 3125/125? [Answer: 25.]
- What is $5^{5-3} = 5^2$? [Answer: 25.]

As you can see, we've checked the specific case of $a^x/a^y = a^{x-y}$ when $x = 5$, $y = 3$, and $a = 5$. Again, we have not proven the general case, but it is better to check twice rather than just once.

This law ($a^x/a^y = a^{x-y}$) also tells us that $8^1 = 8^3/8^2$ because $1 = 3-2$. Then our calculator tells us that $8^3 = 512$ and $8^2 = 64$. So we have

$$\frac{8^3}{8^2} = \frac{512}{64} = 8 \text{ and also } \frac{8^3}{8^2} = 8^{3-2} = 8^1$$

allowing us to conclude that $8^1 = 8$. Similarly $5^1 = 5^3/5^2$ because $1 = 3-2$. Then our calculator tells us that $5^3 = 125$ and $5^2 = 25$. Therefore, we know that

$$\frac{5^3}{5^2} = \frac{125}{25} = 5 \text{ and also } \frac{5^3}{5^2} = 5^{3-2} = 5^1$$

allowing us to conclude that $5^1 = 5$.

You've probably guessed that this is not unique to 8 and 5, but it is true for all numbers. For any positive number a, it is true that $a^1 = a$. This is the sixth law of exponents—a law which does not see frequent use.

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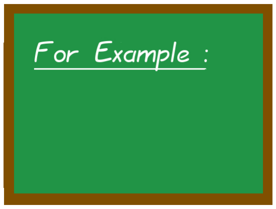


2-7-14

Now, let's consider an application of the above law to finance. If we take the formula for compound interest,

$$A = P(1+i)^n$$

and consider $n = 1$, then we get

$$A = P(1+i)^1 = P(1+i)$$

but this should make sense. We said compound interest problems were like a sequence of simple interest problems, with one for each compounding period. Thus, when there is only one compounding period, then naturally compound interest and simple interest are the same thing!

If we take $A = P(1+rt)$ and plug in $t = 1$ and $r = i$ we would get $A = P(1+i)$, as we have obtained above.

What happens if I do $(x^2)^3$? Well that is going to be $(x^2)(x^2)(x^2)$, because that what cubing does. That's clearly going to be $x^{2+2+2} = x^6$.

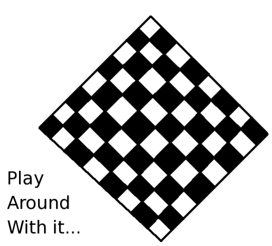
Likewise, what happens if I do $(x^5)^2$? Well that is going to be $(x^5)(x^5)$, because that what squaring does. That's clearly going to be $x^{5+5} = x^{10}$. Again, this can be generalized to

$$(x^a)^b = \underbrace{(x^a)(x^a)\cdots(x^a)}_{\text{b times}} = x^{\overbrace{(a+a+\cdots+a)}^{\text{b times}}} = x^{ab}$$

and therefore

$$(x^a)^b = x^{ab}$$

is the seventh law of exponents.



2-7-15

- What is 3^2? [Answer: 9.]
- What is 9^3? [Answer: 729.]
- What is $3^{2 \cdot 3} = 3^6$? [Answer: 729.]
- What is 4^2? [Answer: 16.]
- What is 16^3? [Answer: 4096.]
- What is $4^{2 \cdot 3} = 4^6$? [Answer: 4096.]

As you can see, we have verified the law $(x^a)^b = x^{ab}$ for the specific cases when $a = 2$, $b = 3$, and either $x = 3$ or $x = 4$.

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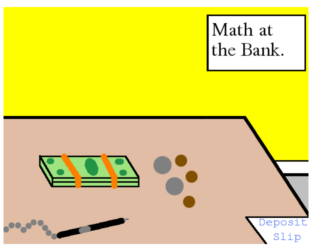
It is very important to understand the difference between $(5^4)(5^6) = 5^{10}$ versus $(5^4)^6 = 5^{24}$. This error is very common, and moreover, the difference between the two numbers is huge. (Just ask your calculator what 5^{10} and 5^{24} are, and you'll see—they're huge.)

$$\begin{aligned} (x^a)^b &= x^{ab} && \leftarrow \text{TRUE !} \\ (x^a)(x^b) &= x^{ab} && \leftarrow \text{FALSE !} \\ (x^a)(x^b) &= x^{a+b} && \leftarrow \text{TRUE !} \end{aligned}$$

Moreover, this error is so common that your instructor has little choice but to put traps of this sort on some examination or another.



Now we're going to see a financial application of some of these laws.

Suppose Bob gets an investment that has a rate of return of 8% per year (compounded monthly), and he invests for 4 years. Meanwhile, Charlie finds one that also has a rate of return of 8% per year (compounded monthly), and invests for one year. He pulls his money out and puts it in Bob's fund for another year, and then decides to switch to his original fund for yet another year, and finally goes back to Bob's fund for the fourth year.

Logic might dictate that both should experience the same return if there are no buy and sell commissions. The reason is that they both had 8% per year (compounded monthly) for 4 years.

In the next box, we'll use some laws of exponents to explore this in more detail.

Continuing with the previous box, suppose they both invest P. Surely Bob would have

$$A = P(1+i)^{mt} = P(1+0.08/12)^{12\times 4} = P(1.006\overline{6})^{48}$$

and we'll leave that unfinished for now.

In comparison, we can call Charlie's return at the end of year 1 as A_1 and at the end of year 2 as A_2, with A_3 and A_4 defined similarly. We'd have $A_1 = P(1.006\overline{6})^{12}$, as well as $A_2 = A_1(1.006\overline{6})^{12}$ and $A_3 = A_2(1.006\overline{6})^{12}$, and lastly $A_4 = A_3(1.006\overline{6})^{12}$. Combining these provides

$$\begin{aligned} A_4 &= A_3(1.006\overline{6})^{12} \\ A_4 &= A_2(1.006\overline{6})^{12}(1.006\overline{6})^{12} \\ A_4 &= A_1(1.006\overline{6})^{12}(1.006\overline{6})^{12}(1.006\overline{6})^{12} \\ A_4 &= P(1.006\overline{6})^{12}(1.006\overline{6})^{12}(1.006\overline{6})^{12}(1.006\overline{6})^{12} \\ A_4 &= P\left[(1.006\overline{6})^{12}\right]^4 \end{aligned}$$

In summary,

$$\text{Bob } = P(1.006\overline{6})^{48} \text{ and Charlie } = P\left[(1.006\overline{6})^{12}\right]^4$$

which might look different symbolically, but we know from the 7th law of exponents that

$$\left[(1.006\overline{6})^{12}\right]^4 = (1.006\overline{6})^{12\times 4} = (1.006\overline{6})^{48}$$

and therefore Bob and Charlie have exactly the same amount, as we anticipated would be the case. Each has $P(1.006\overline{6})^{48} = (1.37566\cdots)P$.

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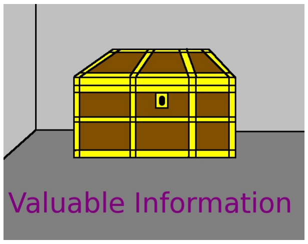


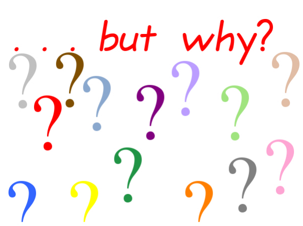
What we've checked in the previous case is $A = P(1+i)^{mt} = P\left((1+i)^m\right)^t$ and that in turn is a special case of $(x^a)^b = x^{ab}$. Particularly, $x = (1+i)$ while $a = m$ and $b = t$ results in

$$(x^a)^b = x^{ab} \qquad \Rightarrow \qquad \left((1+i)^m\right)^t = (1+i)^{mt}$$

and then we get what we require by multiplying both sides of the second equation of the previous line by P.

Believe it or not, this is quite practical. Consider a compound interest problem with a very large n. Suppose we have $r = 7\%$ compounded weekly, for 23 years, and $P = 54,321$. Because it is compounded weekly, I start with $m = 52$, and first ask my calculator to compute $i = 0.07/52 = 0.00134615\cdots$. Then I ask the calculator to add one, and now the parentheses are finished. At this point I would want to raise what is on the calculator's screen to the power 52×23.

Maybe you can reliably compute 52×23 in your head, but I cannot. If I compute 52×23 using the calculator, then I lose what is currently on the calculator's screen, namely $(1+i) = (1+r/m)$, with all its digits of precision. I would also have to spend time copying it either into the calculator's memory or copying it down carefully on to some paper and re-entering it later. Instead, what I should do (with the valuable data still on the screen) is first raise it to the 52nd power, and then raise it to the 23rd power. That's the same thing as raising it only once, but to the 52×23 power. Then I can multiply by $54,321$ and I have finished the problem!



This next law will turn out to be quite practical. We can informally see that

$$\frac{a^3}{b^3} = \frac{aaa}{bbb} = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right)^3$$

but we can also prove it formally.

$$\begin{aligned} a^3/b^3 &= \left(a^3\right)\left(1/b^3\right) \\ &= \left(a^3\right)\left(b^{-3}\right) \text{ by the fourth law} \\ &= (a)^3\left(b^{-1}\right)^3 \text{ by the seventh law} \\ &= \left(a \cdot b^{-1}\right)^3 \text{ by the third law} \\ &= (a \cdot 1/b)^3 \text{ by the fourth law} \\ &= (a/b)^3 \end{aligned}$$

Of course, the rule is not unique to cubes. It could work for any exponent x. Therefore, we'll state the law generally: For any positive numbers a and b, the fraction a^x/b^x is equivalent to $(a/b)^x$. This is the eighth law of exponents.

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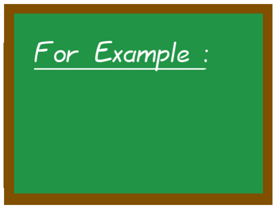


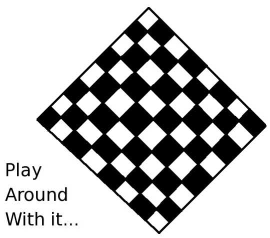
2-7-16

So, at this point, you are probably wondering about the practicality of this law. There will be times when, particularly in the probability and combinatorics part of this book, we must calculate numbers such as $5^{40}/4^{40}$. Well, this can be a bit challenging. The reason is that 5^{40} is an 28 digit number! It definitely will not fit on the display of your calculator. Likewise, 4^{40} is a 25 digit number. However, if we apply the eighth law of exponents, we obtain

$$\frac{5^{40}}{4^{40}} = \left(\frac{5}{4}\right)^{40} = (1.25)^{40} = 7523.16\cdots$$

which really isn't so huge after all!

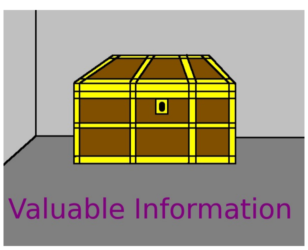
You should *always* use this computational trick when, in later chapters, you have to calculate stuff like $5^{40}/4^{40}$.



2-7-17

Consider finding $2^{27}/3^{27}$.

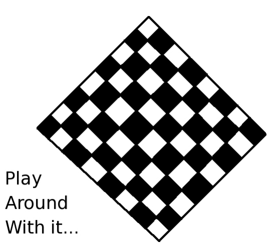
- What is 2^{27}? [Answer: 134,217,728.]
- What is 3^{27}? [Answer: 7,625,597,484,987.]
- What is the first answer divided by the second one? [Answer: 1.76009×10^{-5}.]
- What is $(2/3)^{27}$? [Answer: 1.76009×10^{-5}.]



Do you see how much time this could save you on an exam? The eighth law saves you from having to write down and re-enter these huge numbers, given as our first and second bullets in the previous box. You can jump right to the quantity that you need, as we did in the last bullet of the previous box.

To calculate a^x/b^x you should have your calculator compute $(a/b)^x$, instead.

In fact, depending on your calculator, you might even have to change the first two bullets into 134.217728 and $7,625,597.484987$, essentially changing the units to be "millions," because the original numbers do not even fit into the calculator.



2-7-18

Consider finding $19^{27}/79^{27}$.

- What is 19^{27}? [Answer: $3.36006\cdots \times 10^{34}$.]
- What is 79^{27}? [Answer: $1.72159\cdots \times 10^{51}$.]
- What is the first answer divided by the second one? [Answer: $1.95171\cdots \times 10^{-17}$.]

Note: You really might need to resort to scientific notation when inputting the calculation of the previous bullet.

- What is $(19/79)^{27}$? [Answer: $1.95171\cdots \times 10^{-17}$.]

Note that some calculators will say 0 for the last two answers. That's clearly not the case. There are tricks involving logarithms where you can coax a little extra precision out of your calculator, but we haven't learned about logarithms just yet.

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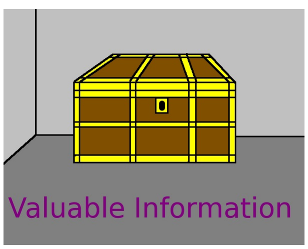
- When we write $5 = \sqrt{25}$ or $7 = \sqrt{49}$, what does the symbol $\sqrt{\ }$ mean? It means that $5 \times 5 = 25$ and that $7 \times 7 = 49$.
- Likewise when we write $2 = \sqrt[3]{8}$ or $3 = \sqrt[3]{27}$ what does the symbol $\sqrt[3]{\ }$ mean? It means that $2 \times 2 \times 2 = 8$ and that $3 \times 3 \times 3 = 27$.
- Now consider $(8^{1/3})^3$. Using the seventh law, we know that this must be $8^{(1/3)\times 3} = 8^1 = 8$.
- So the symbols "$8^{1/3}$" are synonymous with "the thing that you cube to get 8," but that, in turn, is $\sqrt[3]{8}$.
- And so $x^{1/2} = \sqrt{x}$, while $x^{1/3} = \sqrt[3]{x}$, and $x^{1/4} = \sqrt[4]{x}$, *et cetera. . . .*
- Thus, we write the ninth law of exponents, which is that $x^{1/a}$ indicates that number whose ath power is x, which can be abbreviated $x^{1/a} = \sqrt[a]{x}$.



Mathematics has many maneuvers that can be considered move/counter-move. For example, addition and subtraction are opposites, multiplication and division are opposites, and we are now exploring that squaring and square-rooting are opposites, cubing and cube-rooting are opposites, taking the (just as an example) sixth power, and sixth-root are opposites, and so on. This can be summarized by the following list, an expansion of what was found on Page 28, and a list that will be expanded on Page 505 and Page 523. This important topic is called "the theory of inverse functions."

- If $4x = 64$ and you want to "undo" the "times 4", you do $64/4$ to learn $x = 16$.
- If $x/2 = 64$ and you want to "undo" the "divide by 2", you do 64×2 to learn $x = 128$.
- If $x + 13 = 64$ and you want to "undo" the "plus 13", you do $64 - 13$ to learn $x = 51$.
- If $x - 12 = 64$ and you want to "undo" the "minus 12", you do $64 + 12$ to learn $x = 76$.

  To which we now add:

- If $x^2 = 64$ and you want to "undo" the "square", you do $\sqrt{64}$ to learn $x = 8$.
- If $x^3 = 64$ and you want to "undo" the "cube", you do $\sqrt[3]{64}$ to learn $x = 4$.
- If $x^6 = 64$ and you want to "undo" the "sixth power", you do $\sqrt[6]{64}$ to learn $x = 2$.

The collective term for the cube root, 4th root, 5th root, 6th root, and so forth is to call them "the higher roots." You're probably aware that the reason we say

$$\sqrt{36} = 6 \text{ is because } 6^2 = 6 \times 6 = 36.$$

Likewise, for the higher roots:

$$\text{We say } \sqrt[6]{64} = 2 \text{ because } 2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64.$$
$$\text{We say } \sqrt[5]{243} = 3 \text{ because } 3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243.$$
$$\text{We say } \sqrt[4]{256} = 4 \text{ because } 4^4 = 4 \times 4 \times 4 \times 4 = 256.$$
$$\text{We say } \sqrt[3]{125} = 5 \text{ because } 5^3 = 5 \times 5 \times 5 = 125.$$

. . . and if you can bring yourself to understand this, then you understand what the higher roots are all about.

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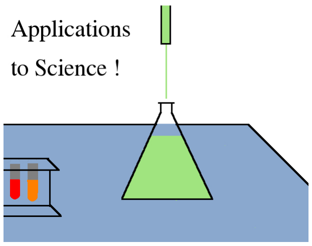
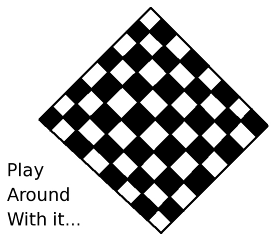


Outside of finance, for example in science and other applied subjects, the higher roots are uncommon. On the one hand, the square root comes up in too many situations to list, and the cube root comes up frequently in problems relating to volumes or three-dimensional figures. On the other hand, the fifth, sixth, and further roots do not occur very often at all, but a notable exception is the important role of the fourth root in radar. We'll also see the 3/2th root and the 2/3rds root in the next module, "Non-Linear Proportions," in relation to Kepler's Laws of planetary motion.

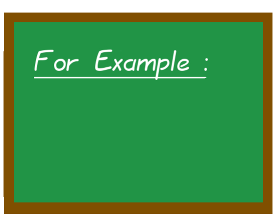
Surprisingly, in finance, the higher roots are indeed very important. For example, we'll require a 5th root on Page 407, a 44th root on Page 462, a 12th root on Page 424, and a 872nd root on Page 351, as well as a 4th root on Page 544. You can flip there and check if you don't believe me.



2-7-19

Without using a calculator, tell me...

- What is the 6th root of one million (also known as 10^6)? [Answer: 10.]
- What is the 4th root of 81? (Note, $81 = 3^4$) [Answer: 3.]
- What is the 45th root of 1.05^{90}? [Answer: 1.05^2.]



2-7-20

Let us consider the following situation. I have \$ 20,000 and I want to save it for the down payment on a house. I need only \$ 25,000 as it turns out. There is a wide array of investment options available to me, with different rates of return. In order to help me choose the right one, what rate of return would be sufficient for me to have the down payment ready 4 years from now? We shall assume that the compounding is monthly.

Naturally $A = 25,000$ and $P = 20,000$. Next, 4 years is $4 \times 12 = 48$ months, so $n = 48$. We now actually have all the information that we need in order to begin calculating. The computation is in the next box.

Continuing with the previous box, we have

$$\begin{aligned}
A &= P(1+i)^n \\
25,000 &= 20,000(1+i)^{48} \\
25,000/20,000 &= (1+i)^{48} \\
1.25 &= (1+i)^{48} \\
(1.25)^{1/48} &= \left((1+i)^{48}\right)^{1/48} \\
\sqrt[48]{1.25} &= (1+i)^{48/48} \\
1.00465\cdots &= 1+i \\
0.00465964\cdots &= i
\end{aligned}$$

and conclude that $i = 0.00465964\cdots$. However, that's per month, so we multiply by 12 and obtain $0.0559157\cdots$ or 5.59%.

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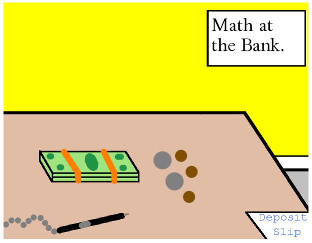
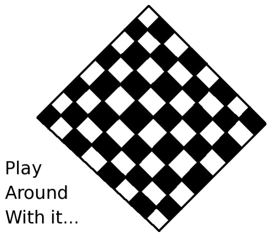
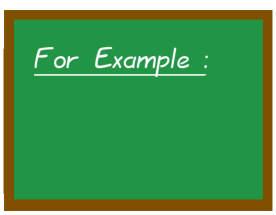


Consider the last box. I have certain financial needs and resources. I was able to calculate what rate of return I require to achieve that need given my resources. Now that I am armed with this information, I can select the lowest-risk investment that returns a rate of return at that rate or higher. It should be possible, except in the worse periods of the economy, to find a bond with this rate, and in fact, a bond with a good credit rating. We will discuss bond ratings later.



2-7-21

I have a certificate of deposit which is coming to maturity in a few days. I bought it 4 years ago, and it compounds quarterly. I will receive \$ 2931.45, and I initially deposited \$ 2500. I have forgotten the interest rate. Can you tell me what it was? [Answer: $i = 1\%$ and $r = 4\%$. Normally, one talks about r, so 4% is what you should write.]



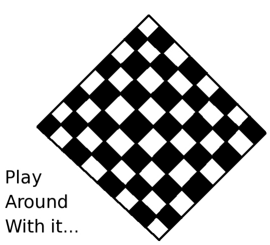
2-7-22

Now we return to the problem about the penny from 1793 that was sold at auction in 2012. This problem was on Page 290, in the module "compound interest." The sales price was 1.38 million dollars, and we are curious to find out what rate of return, compounded quarterly, would produce that kind of profit, from 1793 to 2011.

We start by noting that from 1793 to 2011 is $2011 - 1793 = 218$ years, and 218 years is $218 \times 4 = 872$ quarters of compounding. Then we have the following calculation:

$$\begin{aligned} A &= P(1+i)^n \\ 1.38 \times 10^6 &= 0.01(1+i)^{872} \\ 1.38 \times 10^8 &= (1+i)^{872} \\ \sqrt[872]{1.38 \times 10^8} &= 1+i \\ 1.02172\cdots &= 1+i \\ 0.0217266\cdots &= i \end{aligned}$$

So the $i = 0.0217266\cdots$ quarterly, and multiplying that by 4 we get that $r = 8.69066\cdots\%$. As you can see, it is between 8% and 9%, as predicted on Page 290.



2-7-23

Recently, Bob received a large inheritance of \$ 2,000,000. He wishes to wait three years before he spends it, because he's currently in the military and has limited options, but will retire in three years. What interest rate is required for him to reach \$ 2.5 million at the end of that period? Assume the investment compounds monthly. [Answer: $i = 0.00621768\cdots$ and $r = 0.0746121\cdots$. That's probably not possible with a risk-free investment, but with stocks it would be a below-average rate of return.]

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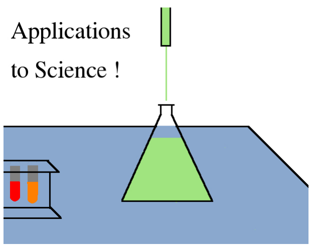
The last things to consider are expressions of the form $x^{25/3}$. Because $25/3 = (25)(1/3)$ then we can think of this as $x^{(25)(1/3)}$ and by the ninth law, we obtain

$$x^{25/3} = x^{(25)(1/3)} = \left(x^{25}\right)^{1/3} = \sqrt[3]{x^{25}}$$

and this works in general. This law will not come up too often in finance, but it is good to know that we can raise any positive number to any fraction. Therefore, we can write

$$a^{b/c} = \sqrt[c]{a^b}$$

and this is the tenth law of exponents.

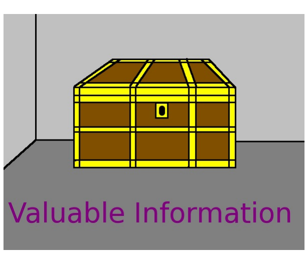


Johannes Kepler (1571–1630) was an astronomer and physicist who is primarily known for discovering three laws of planetary motion, as well as finally working out the mathematics behind the notion that the sun is the center of the solar system.

He was concerned with figuring out how the planets move. The length of time it takes for a planet to go one full lap around the sun is called *the period* of that planet. However, instead of saying "the period of mercury is 88 days", often scientists (and science fiction writers) will say "Mercury's year is 88 days." The reason for this is that the period of the earth is $365.24\cdots$ days, and it is from this that we get most of our calendar (with a 365-day year), the seasons, leap years, and all that. The leap years come about because 365.24 is not an integer.

Kepler's Third Law says that the period of the orbit is proportional to distance$^{(3/2)}$, and also that the distance (from the sun to the planet) is proportional to period$^{(2/3)}$. This is one of those ideas that is much better explained by an example or two.

We will explore this issue of Kepler's Third Law more in the next module, "Non-Linear Proportions." For now, I want you to know that there are genuine effects in science that actually grow as the (3/2)th or (2/3)rds power.



Now, as promised, here are the laws of exponents, all of which we derived carefully:

I. For any positive number a: $a^x a^y = a^{x+y}$.

II. For any positive number a: $a^0 = 1$.

III. For any positive numbers a and b: $(a^x)(b^x) = (ab)^x$.

IV. For any positive number a: $a^{-n} = 1/(a^n)$, and in particular $a^{-1} = 1/a$.

V. For any positive number a: $\frac{a^x}{a^y} = a^{x-y}$.

VI. For any positive number a: $a^1 = a$.

VII. For any positive number a: $(a^x)^y = a^{(xy)}$.

VIII. For any positive numbers a and b: $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$.

IX. Whenever you see $a^{1/b}$, for any positive number a, it indicates that number whose bth power is a, which can be abbreviated $a^{1/b} = \sqrt[b]{a}$.

X. For any positive number a: $a^{b/c} = \sqrt[c]{a^b}$.

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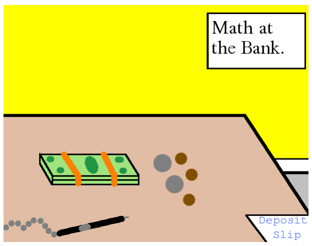
Note, all textbook authors (and all other mathematicians) agree on these laws, but the numbering/organization of them is different in essentially every book. Therefore, remember that these laws are true, but do not focus on which is the third and which is the fifth, and so forth.

I'm now going to show you a real gem of macroeconomics, and one that would be hard to understand without having studied the laws of exponents.



The Cobb-Douglas model is one of the most famous models in economics, and is certainly the most famous one involving strange exponents. This model was published in 1928 by the mathematician Charles Cobb (1875–1949) and the economist Paul Douglas (1892–1976), and was meant to predict the total amount of production in the US economy, as a function of the amount of labor and the amount of capital improvements (investment in future production). This research was based on some data computed from 1899 to 1922, but was later expanded to 1947.

As you can guess by the fact that textbooks still talk about it, the function has other uses. It can be used to model other industrialized nations, entire industries, or even particular companies.

The general model they wanted to design was

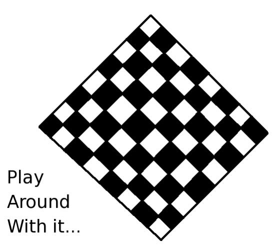
$$P = \beta L^{\alpha} K^{1-\alpha}$$

and the specific fit that they found was

$$P = 1.01 L^{0.75} K^{0.25}$$

where the values mean the following:

- P represents the total monetary value of all that was produced within a year.
- L represents the total amount of labor, meaning the number of work-hours worked in the year (by the entire population).
- K represents the total amount spent on equipment, factories and their machinery, infrastructure, and so forth.
- and as you can see $\alpha = 0.75$ and $\beta = 1.01$ were the parameters that they calculated by analyzing the years 1899–1922.



2-7-24

Using the Cobb-Douglas formula $P = 1.01 L^{0.75} K^{0.25}$ to answer the questions:

- What is P when $L = 110$ and $K = 114$? [Answer: $P = 112.096\cdots$.]
- What is P when $L = 122$ and $K = 131$? [Answer: $P = 125.432\cdots$.]
- What is P when $L = 125$ and $K = 149$? [Answer: $P = 131.916\cdots$.]
- What is P when $L = 140$ and $K = 176$? [Answer: $P = 149.725\cdots$.]

By the way, those data points are the years 1901, 1903, 1905, and 1907, respectively.

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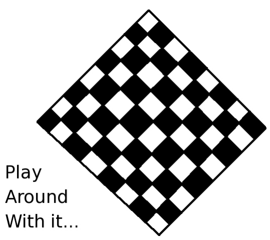
It should be noted what the units are. In any Cobb-Douglas model, a baseline is chosen. In the original data set, the baseline was the year 1899. The total dollar value of production for that year is defined to be $P = 100$. Then for example, in 1907, when we say $P = 149.725$, we mean that $P = 1.49725$ times as much in 1907 as it was in the baseline year of 1899. Similarly, L is the total number of worker-hours of labor used in the economy, with $L = 100$ being the baseline during 1899. Likewise, K is the total dollar value of all capital improvements (Kapital in German) such as factories, equipment, machinery, and infrastructure as compared to the baseline year of 1899 where $K = 100$.



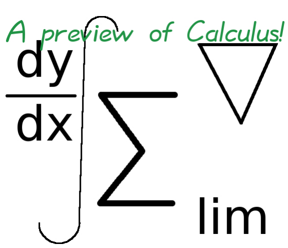
2-7-25

Now let's test the accuracy of the model, using the formula

$$\text{relative error} = \frac{\text{estimate} - \text{truth}}{\text{truth}}$$

- In 1901, the true value of P was 112. What is the relative error?
  [Answer: $0.000857142\cdots$ or 0.08%.]
- In 1903, the true value of P was 124. What is the relative error?
  [Answer: $0.0115483\cdots$ or 1.15%.]
- In 1905, the true value of P was 143. What is the relative error?
  [Answer: $-0.0775104\cdots$ or -7.75%.]
- In 1907, the true value of P was 151. What is the relative error?
  [Answer: $-0.00844370\cdots$ or -0.84%.]

This is rather impressive, really. Something as complex as the entire production system of the US economy is being modeled by a formula that, while certainly not the easiest ever written, is certainly not the hardest ever written either. Despite the anomaly of 1905, the model is overall very accurate.



Later economists proposed better models, and the Cobb-Douglas model itself is (in more recent years) considered a bit too oversimplified. However, it laid the foundations for much modeling that followed, and the equation still appears inside of many textbooks.

An excellent derivation of the model from basic economic principles can be found in James Stewart's *Calculus*, Section 15.3, 6th edition. Of course, one doesn't need to know calculus to work with the Cobb-Douglas equation to some extent (after all, we just used it ourselves) but you can do more with it if you do know calculus.

The coefficients can be calculated by viewing each year's data for (P, L, K) as a point in ordinary 3-dimensional space, and finding the best-fit plane, after an interesting trick involving the taking of a natural logarithm. Looking into these methods is an outstanding problem for someone wanting to explore mathematical economics more deeply, perhaps over a summer, after taking 1–3 semesters of calculus.

Having looked at the very economic topic of the Cobb-Douglas equations, let's look at some more questions in pure mathematics.

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A preview of Calculus!

We've left two questions unanswered. We showed you that you can raise any positive number to any fraction, but you might or might not be aware that certain numbers, like $\sqrt{2}$, can never be written as a fraction. Then it becomes difficult to understand what

$$5^{\sqrt{3}}$$

might mean. However, this is easily explained via "The Squeeze Theorem" in *Calculus I*. Since this book does not cover *Calculus I*, surely I should not explain it here. The matter of 0^0 is more interesting.

A preview of Calculus!

One mathematician might examine the sequence

$$0^{0.1}, 0^{0.01}, 0^{0.001}, 0^{0.0001}, \ldots = 0, 0, 0, 0, \ldots$$

and since the left is clearly going to 0^0, so must the right. The right is clearly going to 0. Then this mathematician would accept that $0^0 = 0$.

Yet another mathematician might examine the sequence

$$0.1^0, 0.01^0, 0.001^0, 0.0001^0, \ldots = 1, 1, 1, 1, \ldots$$

and since the left is clearly going to 0^0, so must the right. The right is clearly going to 1. Then this mathematician would accept that $0^0 = 1$.

This ambiguity cannot be tolerated, and for this reason, 0^0 is listed as one of the "seven indeterminate forms." Like $0/0$, the operation 0^0 is just not defined, and should be avoided at all costs. Do not worry about the seven indeterminate forms, but if you are curious, some others include $\infty - \infty$ and $0 \cdot \infty$, notions that are very hard to understand. Usually this is not explained properly until *Calculus II* or *Calculus III*.

Hard but Valuable!

Up until this point in the module, I have honestly tried to focus on problems which are genuinely economic, scientific, or financial. However, I now have a different sequence of problems for you.

These next few problems are not about economics, science, or finance. They are absolutely pure mathematics. However, only a student who has truly mastered and understood the laws of exponents can solve a problem of this kind. Therefore, these are an outstanding example of calisthenics of the mind.

Some instructors will include this material and others will not. However, I encourage you to do the problems anyway.

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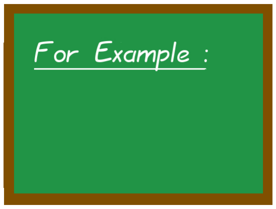


2-7-26

Suppose you had a math problem which had been worked down to

$$\frac{x^2y^{-3}}{x^{-4}y^5}$$

what can you do to finish it?

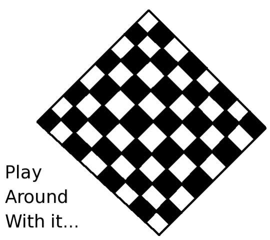
Well we can let the x's battle it out, and the y's also, by grouping the xs together and the ys together as follows:

$$\frac{x^2y^{-3}}{x^{-4}y^5} = \left(\frac{x^2}{x^{-4}}\right)\left(\frac{y^{-3}}{y^5}\right) = \left(x^2x^4\right)\left(\frac{1}{y^3y^5}\right) = \left(x^6\right)\left(\frac{1}{y^8}\right) = \frac{x^6}{y^8}$$

and it may come to pass that you can do this in one step mentally, which is great, but perhaps not, which is fine too. A totally different route would be

$$\frac{x^2y^{-3}}{x^{-4}y^5} = x^2x^{-(-4)}y^{-3}y^{-5} = x^{2+4}y^{-3-5} = x^6y^{-8} = \frac{x^6}{y^8}$$

and that, naturally, produces the same answer. Take any route you wish.



2-7-27

- How would you write

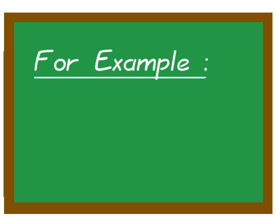
$$\frac{x^6y^2}{x^4y^3}$$

without a fraction bar? Answer $= x^2y^{-1}$.

- How would you write

$$\frac{a^4b^5}{a^6b^3}$$

without a fraction bar? Answer $= a^{-2}b^2$.



2-7-28

Let's say we have the following

$$\frac{5x^{-3}y^{-2}}{25xz^{-1}}$$

which you can see can be helped by getting rid of the negative exponents.

When you send a term to the opposite side of the fraction bar, either from numerator to denominator or from the denominator to the numerator, then the sign of the exponent will flip. As you can see, after that, the problem is easy

$$\frac{5x^{-3}y^{-2}}{25xz^{-1}} = \frac{5z}{25xy^2x^3} = \frac{5z}{25x^4y^2} = \frac{z}{5x^4y^2}$$

Usually math teachers will use the word "simplify" to describe what we just did, and what we are about to do in the next box. I like to avoid this term, because it is vague. What you find simpler, I might not find simpler, and vice versa. Certainly the process which gets us there is not simple.

Therefore, I like to say "Rewrite ______________ where each variable appears once, and only with positive exponents, and no parentheses."

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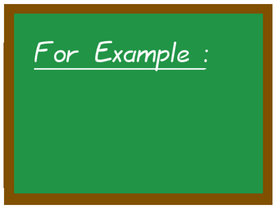


2-7-29

Suppose we must rewrite

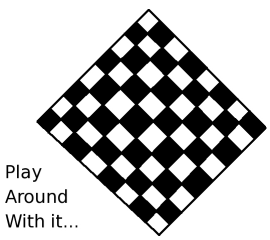
$$\frac{3x^3y^2z^4}{2x^{-2}y^{-5}z^8}$$

where each variable appears once, and only with positive exponents, and no parentheses.

It looks complicated, but the x's only interact with the x's, and the y's only interact with the y's. Likewise, the z's only care about other z's. So you can reorganize it on that basis. The following is excessively detailed, and most students would do some or all of it in their head.

$$\begin{aligned} \frac{3x^3y^2z^4}{2x^{-2}y^{-5}z^8} &= \left(\frac{3}{2}\right)\left(\frac{x^3}{x^{-2}}\right)\left(\frac{y^2}{y^{-5}}\right)\left(\frac{z^4}{z^8}\right) \\ &= (3/2)\left(x^3x^2\right)\left(y^2y^5\right)\left(z^4z^{-8}\right) \\ &= (3/2)(x^5)(y^7)(z^{-4}) \\ &= \frac{3x^5y^7}{2z^4} \end{aligned}$$

Generally, you should ask your instructor how much detail is required. Also be aware that almost every student takes a slightly different route to the one unique solution—and that's just fine.

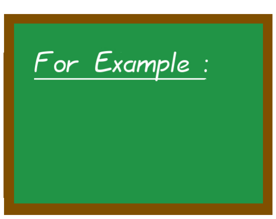


2-7-30

Write the following as (simpler) expressions, where each variable appears once, and only with positive exponents, and no parentheses:

- $\frac{25x^{-1}y^{-4}z^1}{5x^1y^{-5}z^2}$
- $\frac{6x^1y^{-3}z^5}{36x^5y^1z^3}$
- $\frac{32x^2y^3z^0}{8x^{-1}y^5z^2}$

[Answer: The first is $\frac{5y}{x^2z}$, the second is $\frac{z^2}{6x^4y^4}$, and the third is $\frac{4x^3}{y^2z^2}$.]



2-7-31

How can we rewrite

$$\left(\frac{25x^2y^0z^4}{5x^4y^2z^{-2}}\right)^3$$

where each variable appears once, and only with positive exponents, and no parentheses:

First we'd use the 8th law to make it just a ratio of two terms cubed, and then we'd use the 3rd law, to resolve the powers

$$\frac{\left(25x^2y^0z^4\right)^3}{\left(5x^4y^2z^{-2}\right)^3} = \frac{25^3(x^2)^3(y^0)^3(z^4)^3}{5^3(x^4)^3(y^2)^3(z^{-2})^3}$$

and finally the 7th law lets us resolve all of those cubes, then it is the type of problem we've solved before.

$$\frac{25^3x^6y^0z^{12}}{5^3x^{12}y^6z^{-6}} = \frac{(15,625)z^{12}z^6}{125x^{-6}x^{12}y^6} = \frac{125z^{12+6}}{x^{-6+12}y^6} = \frac{125z^{18}}{x^6y^6}$$

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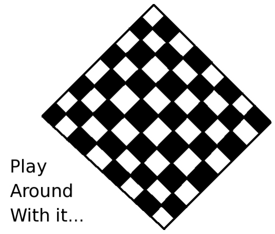


2-7-32

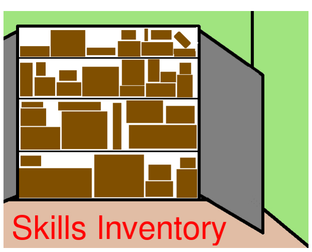
Write the following as simpler expressions, where each variable appears once, and only with positive exponents, and no parentheses:

- $\left(\frac{4xy^4}{16x^{-1}y^{-2}z^5}\right)^2$
- $\left(\frac{9x^{-1}y^{-1}z^{-1}}{81x^{-3}z^5}\right)^{1/2}$
- $\left(\frac{128x^{-3}y^{-3}z^{-2}}{16x^2z^{-1}}\right)^{-4}$

[Answer: The first is $\frac{x^4y^{12}}{16z^{10}}$, the second is $\frac{x}{3z^3\sqrt{y}}$, and the third is $\frac{x^{20}y^{12}z^4}{8^4}$.]

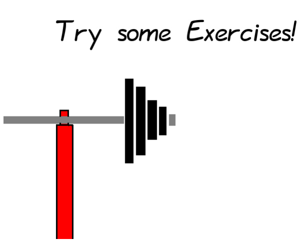
Actually, we made a tiny but silent assumption in the previous box. We assumed that x, y, and z are all positive. In the second bullet, if x were negative, and y and z both positive, we'd get a valid answer, but it would not be $\frac{x}{3z^3\sqrt{y}}$. Instead it would be its negation. For example, if $x = -4$, $y = 1$, and $z = 1$, the second bullet produces $1.3\overline{3} = 4/3$, whereas direct substitution into the answer given produces $-4/3$ for that substitution.

Why is this the case? The short answer is to not worry about it, because in finance, science, and economics, the variables will be positive. The long answer is that $\sqrt{x^2}$ is not actually x as is commonly believed, but is actually $|x|$. Knowing where and when to insert absolute value signs is tedious and takes us well beyond the scope of this book. Any mathematics professor would be delighted to discuss the issue with you during office hours.



We have learned the following skills in this module:

- To use and practice the ten laws of exponents.
- To avoid several algebra pitfalls that stem from misunderstanding the laws of exponents.
- To solve the compound interest formula $A = P(1+i)^n$, when i (or r) is unknown.
- To calculate the volume or weight of a scale model of an object, using cubes.
- To simplify ratios of algebraic expressions using the laws of exponents.
- To handle expressions like $19^{40}/20^{40}$ without calculator overflow, and other calculator tricks.
- We have also seen several applications of exponents in finance, science, and economics, including the Cobb-Douglas formulas.



Coming soon!

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