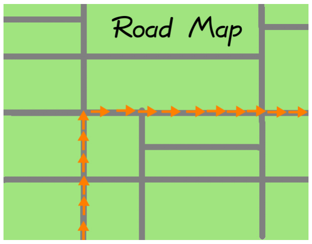
Module 3.7: Radiation, Bacteria, Population, and Real Estate

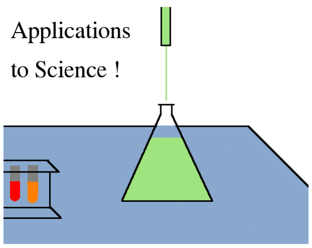


Amazingly, several important but seemingly unrelated concepts—radioactivity, and two common methods of depreciation, as well as the growth in human population or the population of bacteria—all come from the same idea, and that idea is very related to inflation, one of the great forces in the financial realm. The name of that idea is "exponential growth & decay." Here we examine the non-financial examples, inflation and depreciation being covered elsewhere in this textbook.

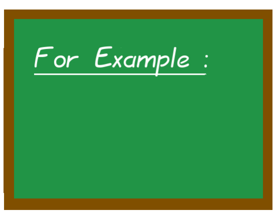
Just as a piece of radioactive material will, over years, give off its energy and lose power slowly over time, so does money lose its purchasing power slowly over time. Likewise, costs of typical items, ranging from the movies to land, will increase steadily over time, as do human populations. All of these formulas a can be represented this way:

$$f(t) = (\text{initial value})\,(\text{some number})^{(\text{some other number} \times t)}$$

In a later module, we will learn how to replace the "some number" that is not in the exponent with the special number "e."



When a piece of radioactive material is examined over time, one sees that it is continuously decaying. Randomly, several of its atoms at any given moment are converting into other types of atoms. The most famous radioactive element is uranium, because of its use in nuclear weapons. Uranium's properties are a bit complicated, but technetium is easier to explain. One type of technetium is Technetium-96, which decays into molybdenum with a *half-life* of 4.3 days—but what is this strange word, "half-life"?



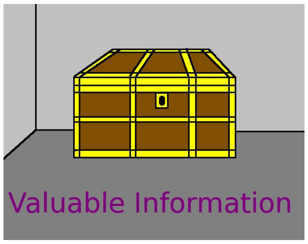
Suppose some atom has a half-life of 10 minutes. This means if you have 800 grams of it, then after 10 minutes, you will $800 \times 1/2 = 400$ grams, but after 20 minutes you will have $800 \times 1/2 \times 1/2 = 200$ grams, and after 30 minutes you will have $800 \times 1/2 \times 1/2 \times 1/2 = 100$ grams. Furthermore, after 40 minutes you will have $800 \times 1/2 \times 1/2 \times 1/2 \times 1/2 = 50$ grams, and after 50 minutes you will have $800 \times 1/2 \times 1/2 \times 1/2 \times 1/2 \times 1/2 = 25$ grams, and so forth. In conclusion, after $10n$ minutes, you will have $A = (800)(1/2)^n$ grams of this substance left.

3-7-1

We can simplify the above process, because it would be tedious to calculate as we did in the previous box. First, what if the time is not a multiple of 10 minutes? Well, if $t = 10n$ then we can divide by 10 to get $t/10 = n$. Then we plug that in to the formula to get $A = (800)(1/2)^{t/10}$. This is easier to write if we remember $(1/2) = 2^{-1}$, using the 4th law of exponents. Using that we would have $A = (800)(2^{-t/10})$ grams.

Next, 10 minutes was specific to our example, but we can call the half-life H. We would obtain then $A = (800)(2^{-t/H})$ grams. Of course, in the above problem, we started out with 800 g, but we could start with any amount A_0. In many branches of knowledge related to mathematics, the little tiny 0 under the variable means "initially" or "when $t = 0$." This symbol is used from physics to economics, from finance to chemistry. Anyway, right now we have $A = A_0(2^{-t/H})$.

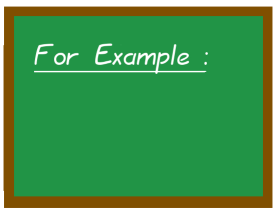
This is an excellent formula, but there's one more short cut. If we have to type this regularly into our calculator, it might be better to rewrite $2^{-t/H}$ or more precisely, $2^{-1/H \times t}$ as instead $(2^{-H})^t$, using the 7th law of exponents. This is used when the value of H is not changing in a problem, but t does change quite often.



Let A_0 be the initial amount of some radio-active substance with half-life equal to H, then after time t, one has

$$A = A_0(2^{-t/H}) \qquad \text{or equivalently} \qquad A = A_0(2^{-H})^t$$

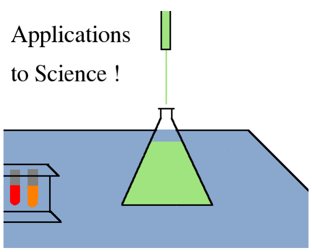
but one must remember that H and t need to be in the same time units (i.e. both seconds, or both years, and so forth).



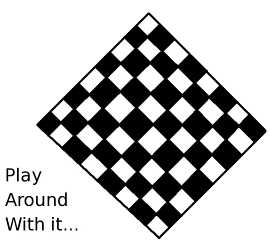
3-7-2

Returning to Technetium-96 we can check an encyclopedia to learn that it has a half life of 4.3 days. If you start with a kilogram of it, how much do you have after 1 day? 2 days? 3 days? 4 days? 5 days? 6 days? 7 days?

$$\begin{aligned}
A(1) &= 1000 \times 2^{-1/4.3} &= 1000 \times 2^{-0.232558\cdots} &= 851.124\cdots \text{ grams}\\
A(2) &= 1000 \times 2^{-2/4.3} &= 1000 \times 2^{-0.465116\cdots} &= 724.412\cdots \text{ grams}\\
A(3) &= 1000 \times 2^{-3/4.3} &= 1000 \times 2^{-0.697674\cdots} &= 616.565\cdots \text{ grams}\\
A(4) &= 1000 \times 2^{-4/4.3} &= 1000 \times 2^{-0.930232\cdots} &= 524.773\cdots \text{ grams}\\
A(5) &= 1000 \times 2^{-5/4.3} &= 1000 \times 2^{-1.16279\cdots} &= 446.647\cdots \text{ grams}\\
A(6) &= 1000 \times 2^{-6/4.3} &= 1000 \times 2^{-1.39534\cdots} &= 380.152\cdots \text{ grams}\\
A(7) &= 1000 \times 2^{-7/4.3} &= 1000 \times 2^{-1.62790\cdots} &= 323.557\cdots \text{ grams}
\end{aligned}$$



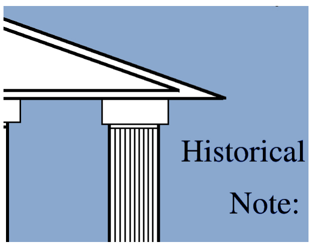
Now remember, the technetium is not vanishing into thin air. In this case, it turns out that it transforms into Molybdenum-96, which is stable. By stable, we mean something that is not radioactive, and just sits there. If we were analyzing something higher up on the periodic table, one atom would decay into another, which decays into another, and then that one decays into another, and the analysis gets more challenging. We'll save that for a proper course in chemistry or physics.



3-7-3

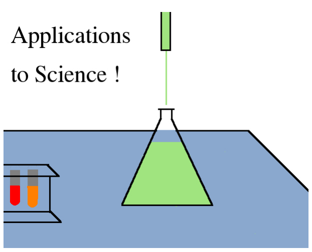
Given the information in the previous box, find the amount after 8 days and after 9 days. [Anwer=$275.387\cdots$ grams and $234.388\cdots$ grams.]

Here's a neat trick for checking your work with half-life problems. We saw that the half-life of technetium was 4.3 days. Thus we expect to have half the sample and a quarter of the sample, or 500 g and 250 g, after 4.3 and 8.6 days, respectively. Now, we were not asked for 4.3 days nor for 8.6 days. Note that 4 days is close to 4.3 days, and 9 days is close to 8.6 days, and we should then check that $A(4) = 524.773$ is close to 500 (and it is) as well as that $A(9) = 234.388$ is close to 250 (and it is). Furthermore, since 4 is an underestimate of 4.3, then the mass at $t = 4$ should be slightly more (and 524.773 is slightly more than 500) and likewise because 9 is an overestimate of 8.6, then the mass at $t = 9$ should be slightly less (and 234.388 is slightly less than 250). Therefore, all is well.

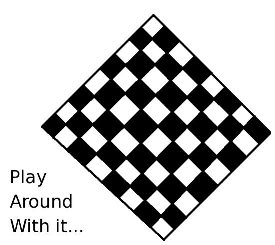


Glenn Seaborg (1912–1999) was a member of the teams that discovered 10 elements of the periodic table, and had one named after him: Seaborgium. He was born in dire poverty, and worked his way through the University California system, eventually becoming the chancellor of UC Berkley. He was on the sub-cabinet of three presidents, as the head of the Atomic Energy Commission, and won a Nobel Prize. His excellent autobiography, *Adventures in the Atomic Age: From Watts to Washington*, describes his rise to prominence, in detail. Once, while discussing some nuclear policy with President George Bush, Sr., he was told that the president's mother was receiving a treatment for thyroid cancer. The isotope used in such tests was Iodine-131, which Seaborg discovered, and he told the president so. Seaborg's life is proof that a scientific career can transform a person born into poverty into a person of power, influence, and fame.

Another more humorous story is that Seaborg's ancestors were Swedes, and he had grown up speaking some Swedish. When he won the Nobel Prize, which is conferred by the King of Sweden, he addressed the crowd in Swedish—only to discover that his family had raised him speaking Swedish with a highly rural and provincial accent. Imagine the surprise of the crowd—hearing a nuclear scientist holding a Nobel Prize speaking like some type of farmer!

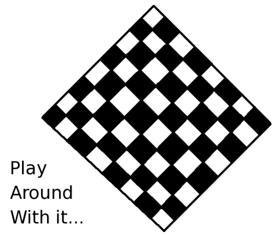


When using radioactive materials in medicine, there is a tradeoff: If the the half-life of the isotope is too short, then all the radiation is released very quickly. The dosage would be very intense, and could easily kill the patient. However, if the half-life is too long, then the substance will stay in the body and possibly have long-term side-effects. If the half-life is very long, then the chemical should be classified as nuclear waste, which is not helpful for marketing. In the particular case of Iodine-131, the human body naturally routes all forms of iodine to the thyroid gland, where it is essential. Thus, Iodine-131 is of excellent use for tests and treatments for thyroid cancer.



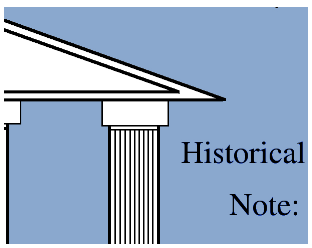
3-7-4

The half-life of Iodine-131 is 8.0197 days. If you are given a dosage of 1000 mg of Iodine-131, how much is left after 1 year? (Use 365 days.) [Answer: $1.9917\cdots \times 10^{-11}$ mg, which is more properly written $19.9179\cdots$ femtograms.] Thus, as you can see, there will be no long term health effects.

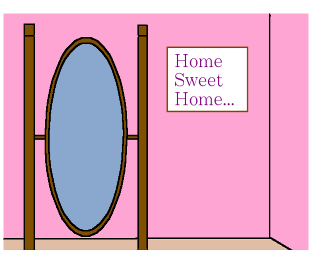


3-7-5

Iodine-129 is one of the atoms that makes up nuclear fall-out, the dust and debris that is left behind after the detonation of an atomic bomb. It is one of seven products termed a "long-term fission product." Unfortunately, the half-life is 15,700,000 years. Furthermore, iodine, being in the same column of the periodic table as fluorine and chlorine, is highly corrosive and reactive, and so this particular atom is the most troublesome for a long-term nuclear waste repository. For comparison the earliest humans (defined here as the earliest members of the *genus homo*) appeared 2.3 to 2.5 million years ago. For comparison, if 30 grams of Iodine-129 is produced in a nuclear weapons blast, how much of it will remain after 1 million years? after 2.5 million years? [Answer: After 1 million years $28.7043\cdots$ grams will remain, and after 2.5 million years $26.8649\cdots$ grams will still remain.]



The United States of America and the Union of Soviet Socialist Republics both conducted a long series of above-ground nuclear test detonations in the 1950s and 1960s. While *The Comprehensive Nuclear Test Ban Treaty* has done away with that now, France, India and Pakistan decided to perform above-ground nuclear-weapons tests of their own in the 1980s and 1990s. Because of these many nuclear explosions, expensive types of scotch, whiskey and wine that claim to be from before the 1950s can be easily tested for radioactive dust. In fact, radioactive isotopes, including strontium in particular, will be found in any scotch made after the first few atomic blasts. In this way, many fraudulent claims about the age of scotch or whiskey have been exposed.

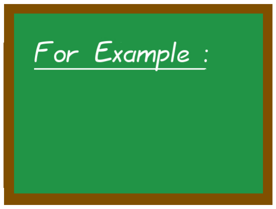


A Pause for Reflection...

The long-term effects of the nuclear tests should give everyone a moment of pause. Decisions made in the 1940–1950 period are still having their effect, long after the decision-makers are dead and buried. While no one can doubt that some in the previous generations made very poor decisions with grave consequences for our future, what decisions are we making now, as a nation, that might have consequences for our not-yet-born grandchildren?

Here's some fodder to get you started:

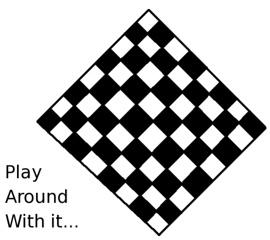
- Pat Ortmeyer and Arjun Makhijani. "Worse than We Knew." *Bulletin of the Atomic Scientists* Nov, 1997.
- Leonard Cole. *Clouds of Secrecy: The Army's Germ-Warfare Tests Over Populated Areas.* Rowman and Littlefield, 1988.
- Mark Wheelis, et al. *Deadly Cultures: Biological Weapons Since 1945*, Harvard University Press, 2006.
- Richard Albright. "Chapter 10: A History of the American University Experiment Station Site." In *Cleanup of Chemical and Explosive Munitions.* William Andrew Inc., 2008.
- Each of these documents should be accessible to anyone with an internet connection.



3-7-6

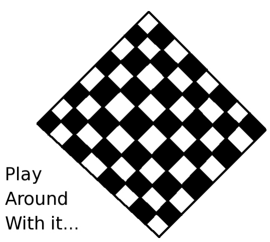
Find A_0 given A, t, and H. Let us suppose some technetium, in particular Technetium-96, has sat for 48 hours and we discover 400 grams of technetium remaining. (The rest is the molybdenum which some other atoms of technetium decayed into.) How much was there originally? Recall that the half-life of Technetium-96 is 4.3 days. Using the formula $A(t) = A_0 2^{-t/H}$ we would have $A(2) = 400 = A_0 2^{-2/4.3}$. Note, I just converted the 48 hours into 2 days. Alternatively, you can write 4.3 days as $4.3 \times 24 = 103.2$ hours. You are obligated to put t and H into the same units. In either case we would then have $400 = A_0 2^{-0.465116\cdots}$ which is $400 = A_0(0.724412\cdots)$ and therefore, $A_0 = 552.171\cdots$ grams.

We can just check our work easily by "forgetting" the 400 grams as $A(2)$, and finding $A(2)$ from $A_0 = 552.171$, as well as $t = 2$ days, and $H = 4.3$ days. Or equivalently, $t = 48$ hours and $H = 103.2$ hours. We would have $A(2) = (552.171\cdots)2^{-2/4.3}$ which is $A(2) = (552.171\cdots)2^{-0.465116\cdots}$. That, in turn, becomes $A(2) = (552.171\cdots)(0.724412\cdots)$ and finally $A(2) = 399.999\cdots$ which is close enough to 400 grams. Another thing to check is by proportion. When A_0 was 1000 grams on Page 475, we saw that 2 days brought us to 724.412. Therefore, $\frac{1000}{724.412} = \frac{552.171}{400}$ should be true. (In other words, the proportion left should be the same.) And it is, therefore, all is well.



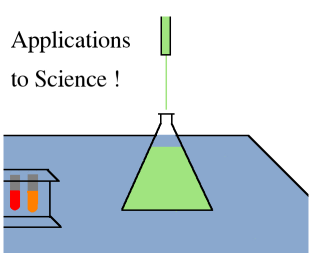
3-7-7

If you have 370 grams of Technetium-96 left over after 8 days, how much did you start with? (Note, the half-life is still 4.3 days.) [Answer: You started with 1343.56 grams.]



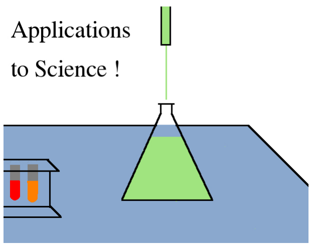
3-7-8

The first atomic bomb blast was at Trinity, New Mexico, on July 16, 1945. The site is now a tourist attraction. Suppose you visit on July 16, 2010, (65 years later) and you find there 9 grams of Technetium-99 left over in some spot. The half-life of Technetium-99 is 211,000 years. How much was there right after the blast? [Answer: $9.00192\cdots$ grams].

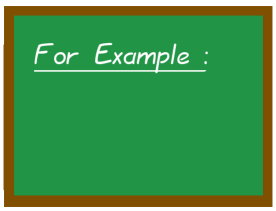


Normally, we think of radioactive decay as a kind of evaporation. Yet, it should be noted that the atoms which are decaying are not vanishing into nothingness, or disappearing. What happens is that each atom, when it decays, becomes some other type of atom. For example, Technetium-96 decays into Molybdenum-96, and Terbium-155 decays into Gadolinium-155. A very tiny amount of mass is annihilated, and this can be found by measuring the energy released as radiation, using Einstein's famous formula $E = mc^2$. The rest of the mass simply becomes the new type of atom.

For example, in the detonation of the nuclear bomb which destroyed Nagasaki, there were 6.2 kilograms of plutonium. Roughly 1 milligram, out of the 6,200,000 milligrams of plutonium, was actually annihilated. The remainder became various forms of nuclear waste, or unexploded plutonium. The radioactive waste resulted in horrific cancer rates for the city over the course of ensuing decades.



One can find on the internet that the dangerous bacteria *E. Coli* doubles every twenty minutes. This means that at any given moment, some of the bacteria in a sample will undergo fission, and divide into two bacteria. Without anything to kill the bacteria, the population should double every twenty minutes. This is a sort of opposite of radioactive decay. In radioactive decay, something might halve every 20 minutes, but here, something doubles every 20 minutes.



3-7-9

Now suppose that you are handling some raw meat, and 3 lonely bacteria end up on your hands. It is not clear that your hands satisfy the status of "ideal conditions" but let us assume that they do. After 10 hours, how many bacteria are there?

The first hour is three twenty-minute intervals, and so that is three doublings, or $2 \times 2 \times 2 = 2^3 = 8$. Then after another hour, one multiplies by 8 again, and another hour is another times 8. Thus, after 10 hours, there are

$$3 \times 8^{10} = 3,221,225,472$$

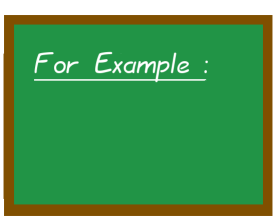
bacteria there. While 3.2 billion bacteria might sound like a lot, each bacteria weighs 7×10^{-13} g and so it it would only weigh $(3.2 \times 10^9)(7 \times 10^{-13}) = 2.24 \times 10^{-3}$ g, or 2.24 milligrams.

Suppose under ideal conditions, the population of bacteria doubles every G minutes. Then after $2G$ minutes one has four times as many, after $3G$, eight times as many, after $4G$ sixteen times as many, and so on. Therefore, after nG minutes, there are 2^n times as many bacteria as one started with, which we can denote P_0. Thus we would write $P = P_0(2^n)$

On the other hand, what about lengths of time that are not multiples of G minutes? If $t = nG$ then $n = t/G$. Therefore, 2^n bacteria can be rewritten as $2^{t/G}$ bacteria. If I start out with P_0 bacteria, then I should have

$$P(t) = P_0 2^{t/G}$$

This can be rewritten by noting $2^{t/G} = (2^{1/G})^t$ and so $P(t) = P_0 \left(2^{1/G}\right)^t$
Also, G is called the *doubling time* of the problem.

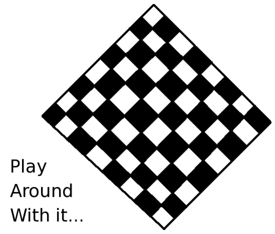


3-7-10

Suppose you have 10^6 bacteria in a Petri dish. You wish to to know, given a doubling time of 20 minutes, how many will be there in 5 minutes, 10 minutes, 15 minutes, 20 minutes, etc..., up to 40 minutes.

We start with $P(t) = P_0 2^{t/20}$, and get

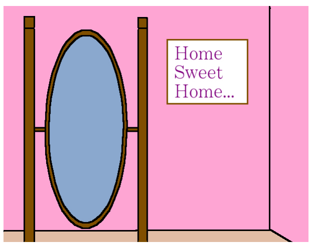
$$\begin{aligned}
P(5) &= 10^6 2^{5/20} &= 10^6(1.18920\cdots) &= 1.18920\cdots \times 10^6\\
P(10) &= 10^6 2^{10/20} &= 10^6(1.41421\cdots) &= 1.41421\cdots \times 10^6\\
P(15) &= 10^6 2^{15/20} &= 10^6(1.68179\cdots) &= 1.68179\cdots \times 10^6\\
P(20) &= 10^6 2^{20/20} &= 10^6(2.00000\cdots) &= 2.00000\cdots \times 10^6\\
P(25) &= 10^6 2^{25/20} &= 10^6(2.37841\cdots) &= 2.37841\cdots \times 10^6\\
P(30) &= 10^6 2^{30/20} &= 10^6(2.82842\cdots) &= 2.82842\cdots \times 10^6\\
P(35) &= 10^6 2^{35/20} &= 10^6(3.36358\cdots) &= 3.36358\cdots \times 10^6\\
P(40) &= 10^6 2^{40/20} &= 10^6(4.00000\cdots) &= 4.00000\cdots \times 10^6
\end{aligned}$$



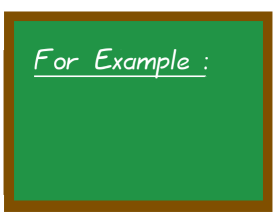
3-7-11

Continue the table by calculating the entries for 45, 50, 55, and 60 minutes.

[Answer: $4.75682\cdots \times 10^6$, $5.65685\cdots \times 10^6$, $6.72717\cdots \times 10^6$, and 8×10^6 bacteria.]



A Pause for Reflection...
Personally, I used to think that all the hygiene rules about how frequently one must wash one's hands were a bit exaggerated—particularly the ones about handling raw meat—but when you realize what three little bacteria can become in a short time, then perhaps they are understandable. For example, in the Math Department at the University of Maryland at College Park, the graduate students when I was there had the "three-second rule" about food. If some article of food were to accidentally fall on the floor, but you grabbed it in less than three seconds, then it didn't count as having fallen on the floor and it was edible. In light of the above problem, perhaps that wasn't very wise.



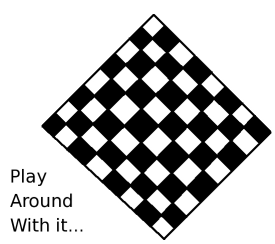
3-7-12

Let us suppose there was a spill of hazardous germs in a lab, around 6 pm on Thursday as the last lab assistant was heading out the door in a hurry to be on time for dinner. Perhaps they were E. Coli. We discover this Friday morning at 6 am, to find that there are roughly a quadrillion bacteria at that time. Assuming ideal conditions, and recalling the doubling time is very roughly 20 minutes, how many bacteria were involved in the initial spill?

Well first we observe that 12 hours went by. That is $12 \times 60 = 720$ minutes. We have

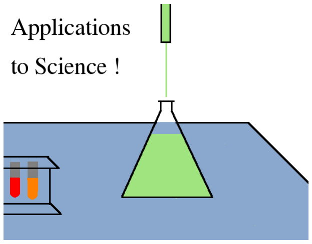
$$\begin{aligned} P(t) &= P_0 2^{t/G} \\ 10^{15} &= P_0 2^{720/20} \\ 10^{15} &= P_0 2^{36} \\ 10^{15} &= P_0 \left(6.87194\cdots \times 10^{10}\right) \\ \frac{10^{15}}{6.87194\cdots \times 10^{10}} &= P_0 \\ 14551.9\cdots &= P_0 \end{aligned}$$

and so roughly 14,552 bacteria were involved in the initial spill.



3-7-13

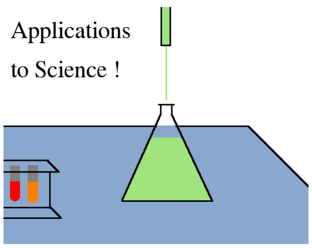
Now suppose that instead of 14,552 bacteria being spilled on Thursday evening, that they were spilled on Friday evening (6 pm), and only discovered on Monday morning at 6 am. Assuming ideal conditions, which is unfair because the bacteria would exhaust the food supply, how many bacteria would be waiting for you when you came to work on Monday morning? Recalling from previous work that they weigh 7×10^{-13} grams each, how much would all the bacteria you find waiting for you weigh? [Answer: $2.23008\cdots \times 10^{58}$ bacteria, weighing $1.56106\cdots \times 10^{46}$ grams.] Just in case you are curious, the earth weighs 5.97223×10^{27} grams and the sun weighs 1.98843×10^{33} grams, so this quantity of bacteria is really impossible, and so something is wrong with the model.



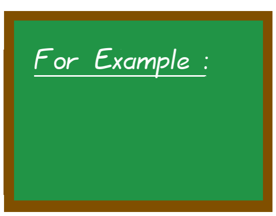
Note: these formulas for bacteria deal with ideal conditions only. A much better model is "the logistic equation," and you can learn about that in one of two books that I recommend:

- You can read, after one good course in calculus, *Introduction to Mathematical Modeling Using Discrete Dynamical Systems* by Frederick R. Marotto.
- Or alternatively, after three semesters of calculus, *Elementary Differential Equations* by William E. Boyce and Richard C. DiPrima.

The logistic equation includes situations like bacteria reproducing to the point that they exhaust the food supply, or where the capacity of the environment is just plain exceeded, and other situations too.



The reproductive habits of bacteria are shared by humans but not most other plants and animals. This relatively odd-sounding statement is due to the fact that most animals and plants breed seasonally. Between breeding seasons, no breeding occurs. In contrast, bacteria will reproduce evenly over any interval of time. Human beings obey the same pattern (with some variation)—both births and deaths are mostly evenly distributed over the course of the year and so the growth in the population is a smooth and continuous pattern.



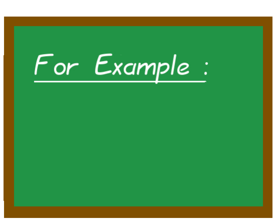
3-7-14

According to *The 2010 CIA World Fact Book*, the human population of the world in July of 2009 was approximately 6,790,062,216. The average birth rate was 19.86 births/1000 population per year , and the death rate was 8.37 deaths/1000 population per year. Thus the growth rate in the population is $19.86 - 8.37 = 11.49$ per thousand per year, or 1.149% per year. Another way to say that is that each year we multiply the current population of the world by 1.01149. Thus, after t years, we multiply the world's population by 1.01149^t.

Thus we can estimate forwards in time:

$$\begin{array}{lrcl}
2010: & 6,790,062,216(1.01149)^1 & = & 6,868,080,030.\cdots \\
2011: & 6,790,062,216(1.01149)^2 & = & 6,946,994,270.\cdots \\
2012: & 6,790,062,216(1.01149)^3 & = & 7,026,815,234.\cdots \\
2013: & 6,790,062,216(1.01149)^4 & = & 7,107,553,341.\cdots \\
2014: & 6,790,062,216(1.01149)^5 & = & 7,189,219,129.\cdots \\
2015: & 6,790,062,216(1.01149)^6 & = & 7,271,823,257.\cdots
\end{array}$$

Just a reminder: our standard of accuracy in our computations in this textbook is six significant figures (unless otherwise stated). That's the 10,000s position in this case. So if you are off by 1000 or 2000 from my numbers, then that's fine.



3-7-15

Continuing with the previous box, you might be curious if we can estimate backwards in time as well, by using negative numbers for t. Actually, we can.

$$\begin{array}{lrcl}
2008: & 6,790,062,216(1.01149)^{-1} & = & 6,712,930,642.\cdots \\
2007: & 6,790,062,216(1.01149)^{-2} & = & 6,636,675,244.\cdots \\
2006: & 6,790,062,216(1.01149)^{-3} & = & 6,561,286,067.\cdots \\
2005: & 6,790,062,216(1.01149)^{-4} & = & 6,486,753,272.\cdots \\
2004: & 6,790,062,216(1.01149)^{-5} & = & 6,413,067,131.\cdots \\
2003: & 6,790,062,216(1.01149)^{-6} & = & 6,340,218,025.\cdots
\end{array}$$

One useful thing is that we can check our model by comparing it with known historical data. Using the formula above, we predict the population in 1999 to have been

$$P_0\,(1+r)^t = 6,790,062,216(1.01149)^{-10} = 6,057,003,414$$

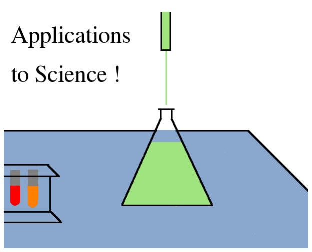
while the 1999 statistic that I was able to find on the Internet says 5,978,000,000. The difference between the prediction and the truth (sometimes called the residual) is $-79,003,414$.

While being off by 79 million sounds like a lot, the relative error is the residual divided by the truth, or the formula:

$$\frac{\text{prediction} - \text{truth}}{\text{truth}} = \frac{6{,}057{,}003{,}414 - 5{,}978{,}000{,}000}{5{,}978{,}000{,}000} = 0.0132156\cdots$$

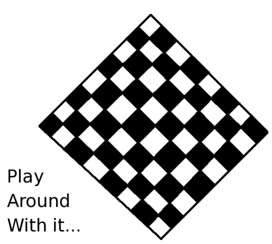
This comes to -1.32%, which is fairly good.

Of course, this means it is really silly to keep more than 4 significant digits in our calculations of the world population, because the real data is not accurate to within 0.01%. However, I am keeping them not because our calculations are very accurate, but so that you can check to see if your calculations match mine.



The population growth of the USA is much smaller. We have 13.83 births/1000 population, much lower than the 19.86 for the whole world overall—so you might not think our population is growing much. On the other hand, our death rate is 8.38 deaths/1000 population, also lower than the world overall, and this comes to a growth rate of $13.83 - 8.38 = 5.45$ per 1000 population per year, or 0.545% per year—very similar to the world's rate. When I first wrote this box, I thought that calculation was complete, but my model did not match data I had found on the Internet.

Then I realized that, unlike modeling the population of the world, when you model the population of a nation, you have to take into account immigration! The number of immigrants minus the number of emigrants is 4.26 per 1000 population per year (that's called the migration rate, and again, it is taken from the *CIA World Factbook*) and so we come to a net change of 9.71 per thousand, or 0.971%, for the USA.



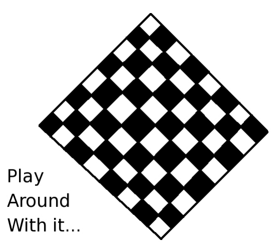
3-7-16

- Using this rate, and the fact that the US population was 307,212,123 in July of 2009, what is the estimated population in 2010? 2011? 2019? Estimate the population in 1999. [Answer: It would be 310,195,152 in 2010; 313,207,147 in 2011; and 338,380,185 in 2019. For 1999, we would estimate 278,914,938.]
- The US population in 1999 turns out to have been 272,639,608 in 1999. Calculate the absolute and relative error of your model. [Answer: The absolute error is $6,275,330$ people, and the relative error is $+2.30\%$, which is not great but not terrible.]

When I had first typed the above example, I had included the births and deaths, but not the migration rate. So I was erroneously using 0.545%, and not 0.971%. This is what I had obtained:

- Using this rate, and the fact that the US population was 307,212,123 in July of 2009, what is the estimated population in 2010? 2011? 2019? Estimate the population in 1999. [Answer: It would be 308,886,429 in 2010; 310,569,860 in 2011; and 324,371,832 in 2019. For 1999, we would estimate 290,960,185.]
- The US population in 1999 turns out to have been 272,639,608 in 1999. Calculate the absolute and relative error of your model. [Answer: The absolute error is $18,320,577$ people, and the relative error is 6.71%, which is rather off the mark.]
- Therefore, as you can see, analyzing the relative and absolute error is a useful tool for detecting when something has gone wrong with your model. If there is some data that is known absolutely, it can be used to test the model.

Before we continue, I'd like to take a moment to mention that populations are never known to the accuracy of plus or minus one person. Usually there is considerable uncertainty, and the population changes every day as people are born and die. However, so that you can verify the correctness of your calculation, I am expressing many more digits of accuracy than I should. This is to help you determine if you are doing the mathematics correctly.



3-7-17

In Japan, things are somewhat different. The birth rate is 7.64 births/1000 people/year, and there are 9.54 deaths/1000 people/year. This means that the population growth rate is $7.64 - 9.54 = -1.90$ per 1000 per year. That's right—it is a negative number. The population is shrinking by -0.19% per year. That is like multiplying by 0.9981 every year. (The net migration is very small, and not easily measured.)

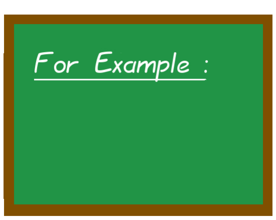
- Using this rate, and the fact that the Japanese population was 127,078,679 in July of 2009, what is the estimated population in 2010? 2011? 2019? Estimate the population in 2002. [Answer: 126,837,229 in 2010; 126,596,238 in 2011; and 124,684,723 in 2019. For 2002, it is 128,781,744.]
- The Japanese population in 2002 turns out to have been 126,974,628. Calculate the absolute and relative error of your model. [Answer: The absolute error is $-1,807,116$ people, and the relative error is -1.42%, not horrible but not all that great.]

The Rev. Thomas Malthus, (1766–1834) was a British philosopher who wrote about topics that would be later called economics. One of his most famous arguments was that populations grow exponentially. Thus, no matter how good agriculture may become, the yield of the earth will eventually be overwhelmed by the population. The growth eventually must be halted, perhaps by disease and famine, or perhaps by war. To make a long story short, he thought the eventual outcome of the human race would be extreme poverty. The "Malthusian argument" is one that says that endlessly large populations are not supportable, because eventually the population will grow to exceed any particular agricultural capacity, predicting an eventual dismal future in a world of starvation.

Malthus had other works as well. His theory of surplus and scarcity would eventually become the theory of supply and demand, though those ideas are quite old, and he also advised parliament on grain importation laws, principally on the basis of his expertise on modeling populations and agricultural production. His prediction of a future world of poverty and starvation was at odds with the standard view of social philosophers, who saw society moving toward some utopian future, making progress sometimes slowly and sometimes quickly, but always moving forward. For this difference he was subjected to attacks by many authors, both during and after his life, including by Karl Marx.

I have a theory about real estate prices. My theory was born when someone who I know quite well bought a nice house in Westchester County, New York. While the house was perfectly adequate, it did not have fancy bathrooms, an up-to-date kitchen, nor a pool or jacuzzi, and not very much land either. The house didn't even have proper air-conditioning except in the master bedroom. In addition, the town is a commuter suburb of New York City, but it is not a very close one; one must ride the train for 40 minutes to reach Manhattan. These are the factors that usually render a house very expensive or not.

Despite these drawbacks, the house cost a small fortune: \$ 667,500, or nearly 2/3rds of a million dollars. My parents remarked that "in Kennedy's day," such a house would have cost far less than that. It would be within the budget of people from humbler walks of life, perhaps costing something like \$ 40,000 to \$ 60,000. Note, JFK was not so long ago, having had the presidency during 1961–1963. Then I hypothesized that real-estate prices feel pressure not just from inflation, but also from the growth in human population.



3-7-18

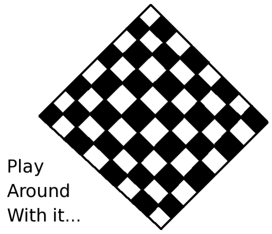
The CPI in 1960 was 29.3, and 50 years later in 2010 it is 216.7. Using the techniques that were taught on Page 456, (in other words just correcting for inflation alone), the \$ 667,500 house would cost in 1960:

$$667,500 \times \frac{29.3}{216.7} = 90,252.65$$

Instead, if we further adjust for the fact that the USA's population in 1960 was 179,323,175, as well as for the fact that a few boxes ago we estimated that in 2010 it was 308,886,429, we would further adjust:

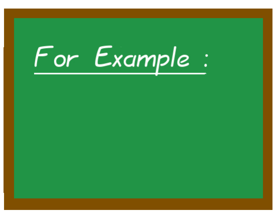
$$90,252.65 \times \frac{179,323,175}{308,886,429} = 52,395.93$$

which is closer to my parent's observation. This effect has a huge impact on "the American Dream," that of owning a nice house with a yard, preferably a house larger than that of the neighbors.



3-7-19

Another year of interest for my parents might be 1972, the year my parents bought their house. At that time, the US Population was roughly 209,896,021, and the CPI was 41.1. How much would the afore mentioned \$ 667,500 house bought in 2010 be worth back then, using my model? [Answer: \$ 86,027.94.]

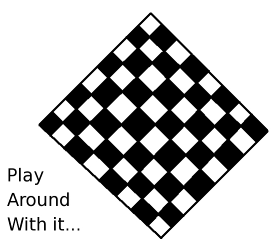


3-7-20

Let us take as a basic forecast that the US Population will continue to grow at the rate of 0.971% per year for the next few decades. Furthermore, let us forecast that inflation will be roughly 4% per year over the next few decades. How much would the previously discussed house be worth, when the mortgage is paid off 30 years later, under my model, given that it was purchased in 2010 for \$ 667,500?

The notion is that we adjust both for inflation as well as population growth, for 30 years. Then we would obtain

$$(667,500)(1+0.00971)^{30}(1+0.04)^{30} = (667,500)(1.33628\cdots)(3.24339\cdots) = 2,893,018.24$$



3-7-21

- Another source of data says that the population growth will be around 1.3% and inflation will be around 4.5%. Then what price should I expect in 30 years? What if I have a 15-year mortgage instead, and want to know what will happen 15 years from now? [Answer: \$ 3,683,183.39 for 30 years, and \$ 1,567,968.40 for 15 years.]
- Of course, there's no justification for this level of precision, because we have a lot of uncertainty about those growth rates. The digits are there to help you check your calculation.

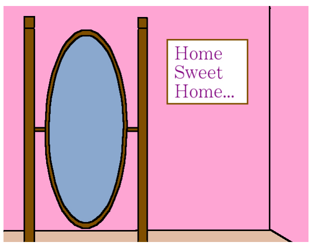
Here, I have made some simplifications to my model. One should really use local versus national population changes, as different parts of the country grow and shrink at different rates. For example, Troy, NY as well as Schenectady, NY have experienced an exodus of population continuously over the last 30-40 years, and correspondingly, real estate is cheap there. Meanwhile, Atlanta, GA and Las Vegas, NV experienced huge population booms in the late 1990s and early 2000s, and the price of real estate did sky rocket there. Thus the best population figures to work with would be those of the metropolitan area in which the real-estate is located, taken to be a broad circle including the suburbs.

The discussion will continue in the next box.

Continuing with the previous box, the Census Bureau has several such designations: “Combined Statistical Area” (or CSA), “Core Based Statistical Area” (or CBSA), “Metropolitan Statistical Area” (or MSA), and “Micropolitan Statistical Area” (or μSA). It would be an interesting undergraduate research project to see which is the best predictor.

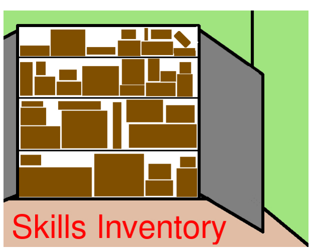
Other factors include changes in how many people live in each household. Families used to be larger, but recently the birth rate has been declining. This means that the same number of persons represents a larger number of required habitations. This because the same number of people indicates a larger number of families since the size of each family is smaller.

Another factor includes investment properties: when persons buy a house for the intent of not living in it, but rather, renting it out. This adds another type of demand on a property, and furthermore, the renter must pay significantly more, because one extra person must draw profits from the cost of his or her housing. As you can see, there is much work that can be done to create accurate models of housing prices.



A Pause for Reflection. . .

We have now learned about exponential functions, and we have seen how they measure and can model effects as diverse as radioactivity, human population, bacteria and real estate. We have tested our models, and seen that while perfect accuracy is often unattainable, one can also often do relatively well, within a few percentage points. We will learn on Page 1345 how to convert all of these models to a universal notation, using the special number e.



We have learned the following skills in this module:

- To identify an exponential model.
- To model radioactivity, and solve for the amount present at any point.
- To model the growth of populations of humans and bacteria.
- To critique exponential models, such as exponential growth of bacteria over a long time, or my real estate model.
- As well as the vocabulary terms: half-life, and doubling time.