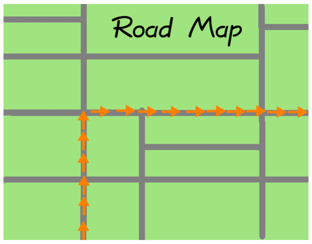
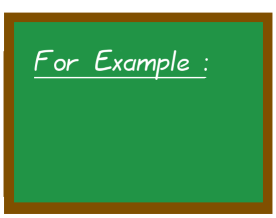
Module 7.3: Intermediate Venn Diagram Problems



In this module, we will explore relationships between set theory and mathematical logic. We will use shaded Venn Diagrams to discern whether some pair of set theoretical expressions is equivalent or not equivalent. We will practice translating English language sentences into set theory, and vice-versa. This will help us improve our understanding of set theory. Moreover, it will also give us some insight into how mathematical logic and set theory can come up inside of databases, including in some commercial, industrial, and financial situations.

One of the things we can do with Venn Diagrams, instead of counting, is to allow the Venn Diagram to help us understand what a set-theoretic expression is trying to say, or to determine if two set theory expressions are saying the same thing.

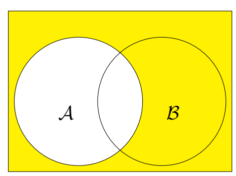
In these Venn Diagrams, we will display a subset. Those regions of the Venn Diagram included in the subset are shaded yellow, and those that are excluded are shaded white. This is a visual representation of the subset.



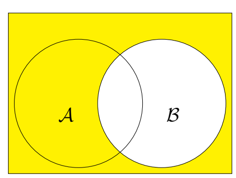
\# 7-3-1

Consider $\mathcal{A}^c \cap \mathcal{B}^c$. What does that sequence of symbols really mean? Let's construct a Venn Diagram with shading to show that set.

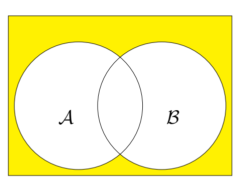
First, we know that $\mathcal{A}^c$ is everything other than $\mathcal{A}$, and so we draw that as:

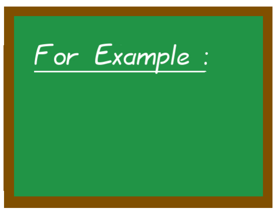


Then, we know that $\mathcal{B}^c$ is everything other than $\mathcal{B}$, and so we draw that as:



The last thing we need to do is to carry out the intersection, which would be to keep as shaded only those regions which are shaded in both the above diagrams.

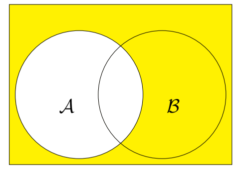




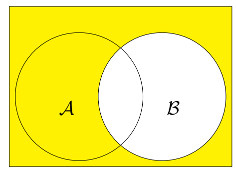
7-3-2

Because the previous box might have been confusing, we'll try a parallel example. Let's try this with $\mathcal{A}^c \cup \mathcal{B}^c$, where we have changed the intersection into a union.

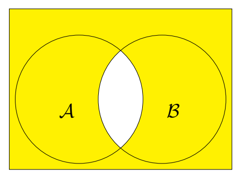
First, we know that $\mathcal{A}^c$ is everything other than $\mathcal{A}$, and so we draw that as:

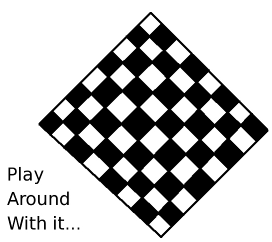


Then, we know that $\mathcal{B}^c$ is everything other than $\mathcal{B}$, and so we draw that as:



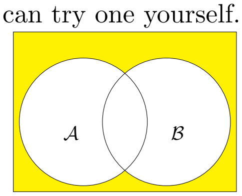
The last thing we need to do is to carry out the union, which would be to keep as shaded any region which is shaded in either of the above diagrams.





7-3-3

Now you can try one yourself. How about this: $(\mathcal{A} \cup \mathcal{B})^c$?

[Answer: 

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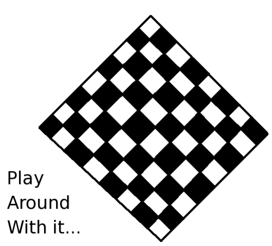
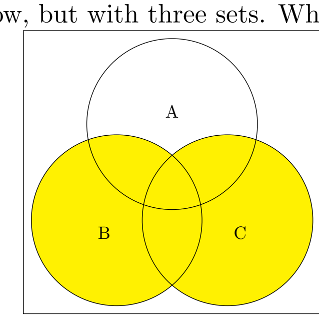


7-3-4

Try it now, but with three sets. What is $\mathcal{B} \cup \mathcal{C}$?

[Answer: 

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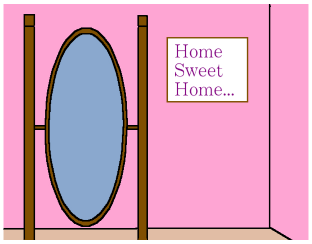
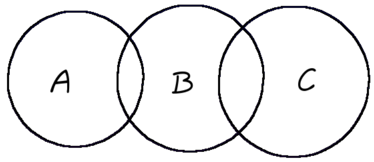
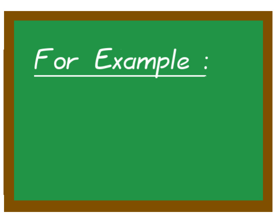


A Pause for Reflection...
I sometimes get questions from students about why 3-circle Venn Diagrams look the way that they do. Some students find the interlocking structure of the three circles to be unnecessarily complicated. Often, they propose something like the diagram below, which is simpler.



The difficulty is that the diagram which they suggest cannot show $\mathcal{A} \cap \mathcal{C}$, because the circle for $\mathcal{A}$ and the circle for $\mathcal{C}$ do not intersect at all, in that picture.

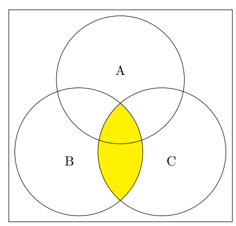
In any case, even if the standard 3-circle Venn Diagram looks complicated now, rest assured that you will be very accustomed to it by the time that you complete this module and the next module.



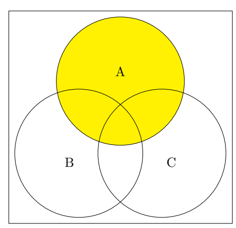
7-3-5

Let's try a harder one with three sets. How about $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C})$?

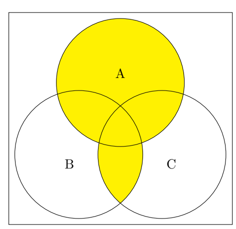
By now we can just draw $\mathcal{B} \cap \mathcal{C}$ directly:



We know what $\mathcal{A}$ is



As before, the last thing we need to do is to carry out the union. The way to do this is to shade any region which is shaded in either of the above diagrams.



Play Around With it...

7-3-6

Try it now yourself:

1. What is $\mathcal{A} \cup \mathcal{B}$?
2. What is $\mathcal{A} \cup \mathcal{C}$?
3. What is $(\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$?

The answer is given on Page 883.

Play Around With it...

7-3-7

How about these?

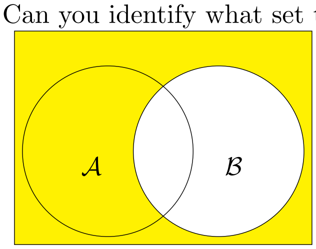
1. What is $\mathcal{A} \cap (\mathcal{B} \oplus \mathcal{C})$?
2. What is $\mathcal{A} \cup (\mathcal{B} \oplus \mathcal{C})$?

The answer is given on Page 883.

Play Around With it...

7-3-8

Can you identify what set theory formula would make this?

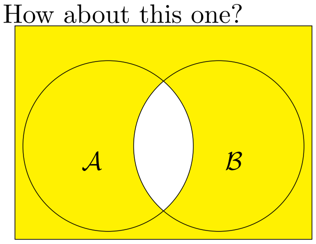


[Answer: $\mathcal{B}^c$.]

Play Around With it...

7-3-9

How about this one?

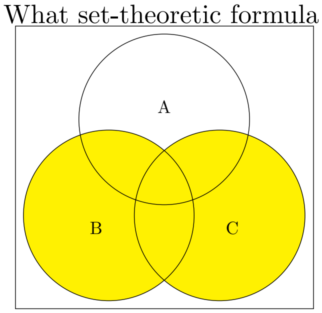


[Answer: $\mathcal{A}^c \cup \mathcal{B}^c$.]

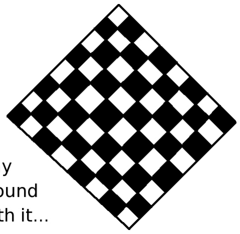
Play Around With it...

7-3-10

What set-theoretic formula would make this?



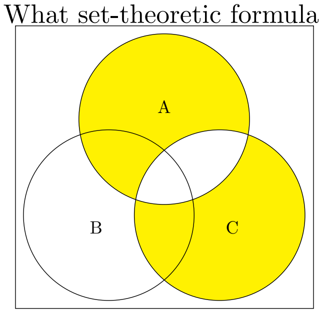
[Answer: $\mathcal{B} \cup \mathcal{C}$.]



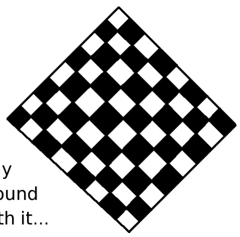
Play
Around
With it...

7-3-11

What set-theoretic formula would make this?



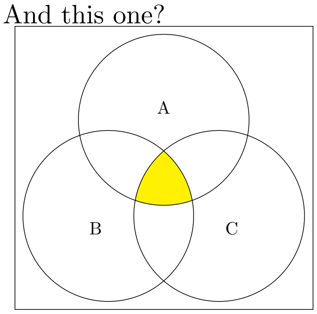
[Answer: $\mathcal{A} \oplus \mathcal{C}$.]



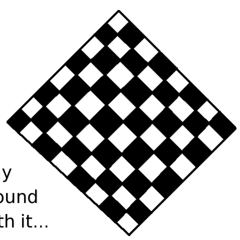
Play
Around
With it...

7-3-12

And this one?



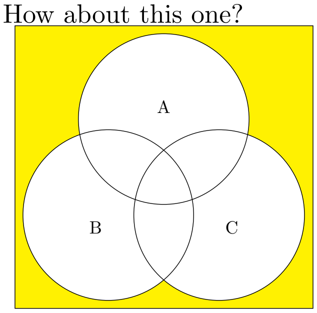
[Answer: $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$.]



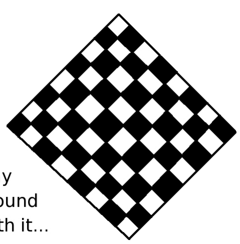
Play
Around
With it...

7-3-13

How about this one?



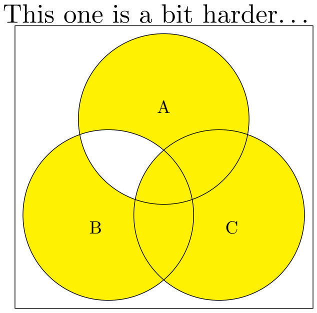
[Answer: $\mathcal{A}^c \cap \mathcal{B}^c \cap \mathcal{C}^c$.]



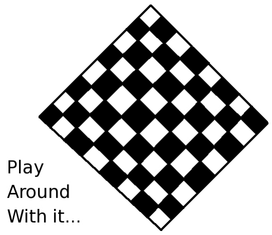
Play
Around
With it...

7-3-14

This one is a bit harder...

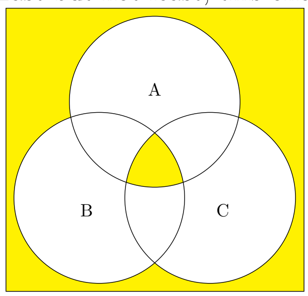


[Answer: $(\mathcal{A} \oplus \mathcal{B}) \cup \mathcal{C}$.]



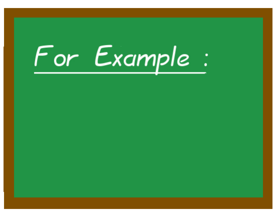
7-3-15

Last but not least, this one?



[Answer: $(\mathcal{A}^c \cap \mathcal{B}^c \cap \mathcal{C}^c) \cup (\mathcal{A} \cap \mathcal{B} \cap \mathcal{C})$.]

Note, there are many other possible answers to the previous box, but I've listed the one that students usually discover.



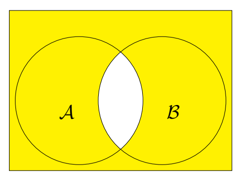
7-3-16

One of the uses of this shading idea, in addition to understanding what a set-theoretic expression is trying to say, is determining if two set-theoretic expressions are equal. We're going to start by examining

$$\mathcal{A}^c \cup \mathcal{B}^c = (\mathcal{A} \cap \mathcal{B})^c$$

and seeing if the two expressions on either side of the = are really equal or not.

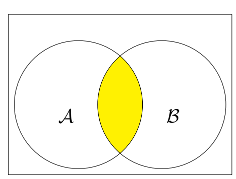
Earlier, on Page 859 we found out that $\mathcal{A}^c \cup \mathcal{B}^c$ has a Venn Diagram of



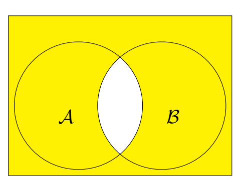
and similarly, we should now calculate what $(\mathcal{A} \cap \mathcal{B})^c$ will have as its Venn Diagram.

In continuation of the previous box, we are now going to calculate the Venn Diagram for $(\mathcal{A} \cap \mathcal{B})^c$.

Because $\mathcal{A} \cap \mathcal{B}$ has a Venn Diagram of

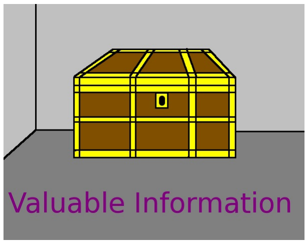


we therefore know $(\mathcal{A} \cap \mathcal{B})^c$ has a Venn Diagram of



and note that all I did was flip the colors, to show the complement.

Now, compare the diagrams! That last diagram is the same as what we calculated earlier, on Page 859.

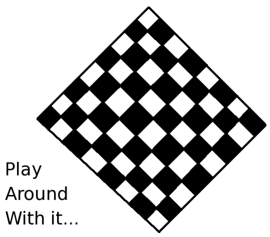


Because the Venn Diagrams for $(\mathcal{A} \cap \mathcal{B})^c$ (previous box) and for $\mathcal{A}^c \cup \mathcal{B}^c$ (two boxes ago) are identical, we can safely conclude that

$$\mathcal{A}^c \cup \mathcal{B}^c = (\mathcal{A} \cap \mathcal{B})^c$$

This fact is half of "DeMorgan's Law of Sets."

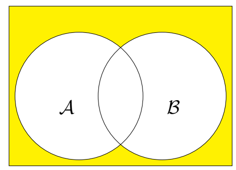
Now you shall find the other half of DeMorgan's Law yourself.



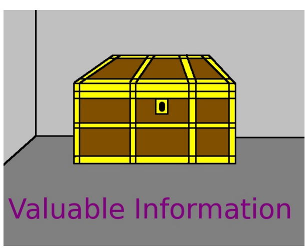
7-3-17

Let us now try the following.

- Earlier, we found out, on Page 857 that $(\mathcal{A} \cup \mathcal{B})^c$ has a Venn Diagram of



- Next, find the Venn Diagram of $\mathcal{A}^c \cap \mathcal{B}^c$.
- They should match! This allows you to conclude that $\mathcal{A}^c \cap \mathcal{B}^c = (\mathcal{A} \cup \mathcal{B})^c$.



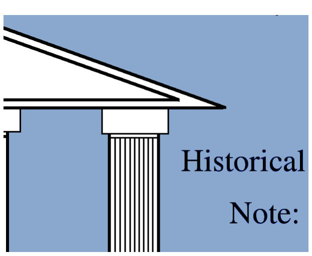
The following two equalities, taken together, are called *DeMorgan's Law of Sets*.

$$\mathcal{A}^c \cap \mathcal{B}^c = (\mathcal{A} \cup \mathcal{B})^c \qquad \text{and also} \qquad \mathcal{A}^c \cup \mathcal{B}^c = (\mathcal{A} \cap \mathcal{B})^c$$

Many students remember this by looking at the right-hand side. When you pull the ^{c} into the parentheses, you need remember to do only two things:

- First, you must remember to give that ^{c} to each of the sets inside.
- Second, you must remember to turn the $\cup$ or $\cap$ upside-down.
- In a certain sense, you are complementing not only each set, but you are also complementing the operator $\cap$ or $\cup$.

If you try to learn this rule in a visual way, you might find it very easy to remember.



The life and career of Augustus De Morgan (1806–1871) is fascinating; his mathematics, somewhat less so. Mathematically, he worked mostly on logic and what modern mathematicians call "foundations." Out of all the branches of math, the foundations movement is probably the view of mathematics most alien to this textbook. The idea is to find a very small set of axioms from which all of mathematics can be proven, and proven very rigorously. This is an insurmountable task, but De Morgan tried to primarily focus on the theorems of algebra. He was not successful but his attempts were very influential on the future of pure mathematics.

He began his university career at Cambridge and did well in mathematics as an undergraduate, and was a popular flute player. He came in 4th place ("fourth wrangler") in the mathematics competition for all the graduating seniors in all colleges that comprise Cambridge University.

Historical Note:

Continuing with the previous box, despite his significant achievements, he could not attempt a graduate degree without passing a theological test. That test was designed to ensure that all graduate students were familiar with and agreed with the teachings of The Church of England.

DeMorgan was morally opposed to such a test, and therefore refused to take it. This left him with very few career prospects. Luckily for DeMorgan, there were other people similarly inconvenienced.

It was for this reason that the university which is now called University College London was founded in 1826. After all, Oxford and Cambridge were still, at that time, entirely unwilling to consider hiring anyone who was unable or unwilling to pass the theological test. At the age of 22, DeMorgan was UCL's first math professor. The theological requirement was still enforced until shortly after DeMorgan died.

Historical Note:

Continuing with the previous box, DeMorgan and his son founded the London Mathematical Society, which today is very important in the mathematical community. DeMorgan also wrote encyclopedia articles, contributing about 1/6th of the articles in *The Penny Cyclopaedia*, published by "The Society for the Diffusion of Useful Knowledge." It was a very cheap 30-volume encyclopedia, published in installments around 1828–1843, designed to disseminate knowledge to all sorts of readers who could not afford more expensive forms of education.

Play Around With it...

7-3-18

- Find the Venn Diagram of $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C})$. Note, you already did this on Page 858.
- Find the Venn Diagram of $(\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$. Note, you already did this on Page 859, as Part 3 of a three-part checkerboard. (The answer is to be found on Page 883.)
- They should be the same. Therefore, using these two diagrams, you can conclude that

$$\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$$

Play Around With it...

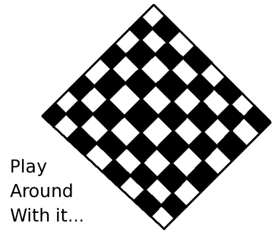
7-3-19

To prove a further point, let's consider three sets of numbers.

$$\begin{aligned} \mathcal{A} &= \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\} \\ \mathcal{B} &= \{3, 6, 9, 12, 15, 18\} \\ \mathcal{C} &= \{5, 10, 15, 20\} \end{aligned}$$

- What is $\mathcal{B} \cup \mathcal{C}$? [Answer: $\{3, 5, 6, 9, 10, 12, 15, 18, 20\}$.]
- What is $\mathcal{A} \cap \mathcal{B}$? [Answer: $\{6, 12, 18\}$.]
- What is $\mathcal{A} \oplus \mathcal{C}$? [Answer: $\{2, 4, 5, 6, 8, 12, 14, 15, 16, 18\}$.]

We will continue in the next box.



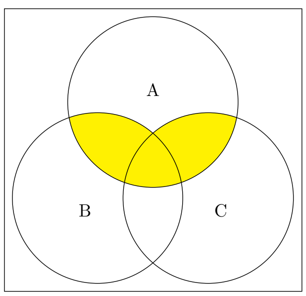
7-3-20

Continuing with the previous box,

- What is $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C})$? [Answer: $\{6, 10, 12, 18, 20\}$.]
- What is $(\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C}$? [Answer: $\{5, 6, 10, 12, 15, 18, 20\}$.]
- For this example, is the following statement true? $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C}$ [Answer: No.]
- In general, is the following rule true? $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C}$

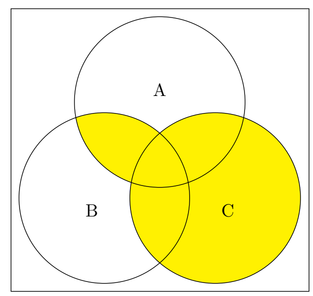
  [Answer: No! Since it didn't work for the particular example in this box, obviously the rule cannot be true in general.]

Here's a representation of the above work, using Venn Diagrams.

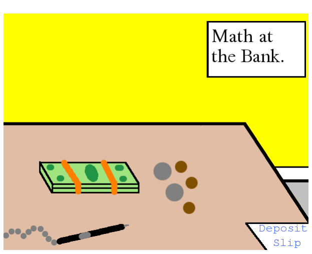


$\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C})$

$\neq$
$\neq$



$(\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C}$



Okay, at this point, we have determined that

$$A \cap (B \cup C) \neq (A \cap B) \cup C$$

is a fact, but it took some effort to accomplish this. You might be wondering if this is useful, or if it is merely calisthenics of the mind.

For an application of having determined this fact, imagine an IT company that sells certifications in some programming language. It would be ambiguous, and therefore very unwise, if they were to write: "To get this certification, you have to pass Test 1 and Test 2 or Test 3." because that could mean $T_1 \cap (T_2 \cup T_3)$ or it could mean $(T_1 \cap T_2) \cup T_3$.

Since we know that this is a problem, we know that they should write either:

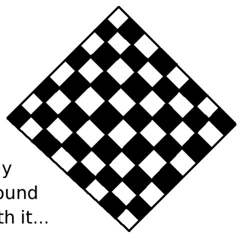
- To get this certification, you have to pass either Test 2 or Test 3, and also Test 1. That's correct if they mean $T_1 \cap (T_2 \cup T_3)$.
- To get this certification, you have to either pass both Test 1 and Test 2, or alternatively pass Test 3. That's correct if they mean $(T_1 \cap T_2) \cup T_3$.

Writing an ambiguous statement could result in a law suit. For example, if the first bullet above is what is intended, namely $T_1 \cap (T_2 \cup T_3)$, then a student who has taken only T_3 but taken neither T_1 nor T_2 might sue when the certification is denied, because that student thought the rule was $(T_1 \cap T_2) \cup T_3$.

In general, we should be very careful to always use parentheses when we have $\cap$s and $\cup$s together in a set-theoretic expression. By looking over student exams through the years, I've found that students often neglect the importance of parentheses, and have the following misconception:

$$\text{WRONG!} \rightarrow \qquad \mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C} \qquad \leftarrow \text{WRONG!}$$

At this point, I think we've firmly established that the parentheses are absolutely necessary. We will explore this point further on Page 866.



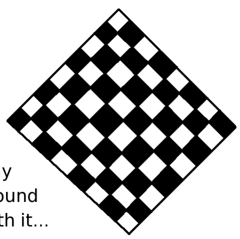
Play
Around
With it...

7-3-21

- Find the Venn Diagram of $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C})$.
- Find the Venn Diagram of $(\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$.
- Once again, they should be the same. As before, you can conclude that

$$\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$$

The diagram that is the answer to both questions is given on Page 883.



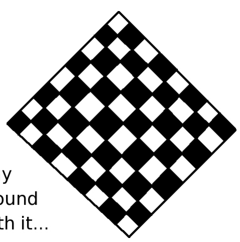
Play
Around
With it...

7-3-22

- Find the Venn Diagram of $\mathcal{A} \cap (\mathcal{B} \cap \mathcal{C})$.
- Find the Venn Diagram of $(\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C}$.
- The Venn Diagrams should be the same. For sure, you can conclude that

$$\mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C}$$

The diagram that is the answer to both questions is given on Page 883.



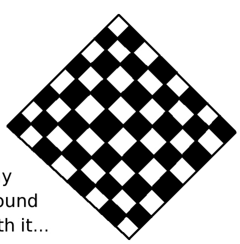
Play
Around
With it...

7-3-23

- Find the Venn Diagram of $\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C})$.
- Find the Venn Diagram of $(\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}$.
- Again, the diagrams should be identical. Therefore, you can conclude that

$$\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}$$

The diagram that is the answer to both questions is given on Page 884.



Play
Around
With it...

7-3-24

- Find the Venn Diagram of $\mathcal{A} \oplus (\mathcal{B} \oplus \mathcal{C})$.
- Find the Venn Diagram of $(\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{C}$.
- Again, the diagrams should be identical. Therefore, you can conclude that

$$\mathcal{A} \oplus (\mathcal{B} \oplus \mathcal{C}) = (\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{C}$$

The diagram that is the answer to both questions is given on Page 884.

Earlier (on Page 865), we established that

$$\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) \neq (\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C}$$

Only if a single operator is present can the parentheses be dispensed with. For example,

$$\mathcal{W} \cup \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$$

unambiguously means anything found in any of the four sets $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, or $\mathcal{Z}$, whereas

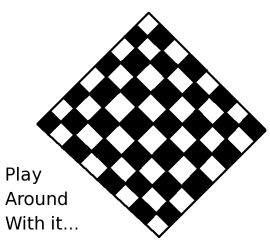
$$\mathcal{W} \cap \mathcal{X} \cap \mathcal{Y} \cap \mathcal{Z}$$

unambiguously means anything found in each of the four sets $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$.

Since there is only one operator in those last two cases, you can dispense with the parentheses. This also works for $\oplus$ in the sense that

$$\mathcal{X} \oplus (\mathcal{Y} \oplus \mathcal{Z}) = (\mathcal{X} \oplus \mathcal{Y}) \oplus \mathcal{Z}$$

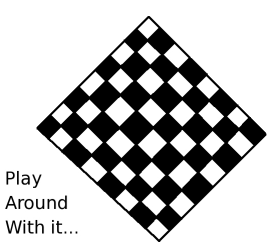
but the English-language meaning of that expression is not so easy to summarize in words.



7-3-25

- Find the Venn Diagram of $\mathcal{A} \oplus (\mathcal{B} \cup \mathcal{C})$.
- Find the Venn Diagram of $(\mathcal{A} \oplus \mathcal{B}) \cup \mathcal{C}$.
- Are the diagrams identical? [Answer: No.]
- In general, is it true that $\mathcal{A} \oplus (\mathcal{B} \cup \mathcal{C}) \neq (\mathcal{A} \oplus \mathcal{B}) \cup \mathcal{C}$? [Answer: Absolutely not!]

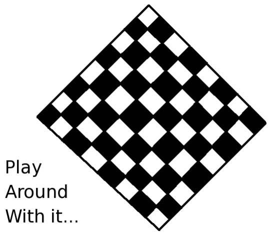
The diagrams for $\mathcal{A} \oplus (\mathcal{B} \cup \mathcal{C})$ and $(\mathcal{A} \oplus \mathcal{B}) \cup \mathcal{C}$ are given on Page 884.



7-3-26

- Find the Venn Diagram of $\mathcal{A} \cap (\mathcal{B} \oplus \mathcal{C})$.
- Find the Venn Diagram of $(\mathcal{A} \cap \mathcal{B}) \oplus \mathcal{C}$.
- Are the diagrams identical? [Answer: No.]
- In general, is it true that $\mathcal{A} \cap (\mathcal{B} \oplus \mathcal{C}) \neq (\mathcal{A} \cap \mathcal{B}) \oplus \mathcal{C}$? [Answer: Absolutely not!]

The diagrams for $\mathcal{A} \cap (\mathcal{B} \oplus \mathcal{C})$ and $(\mathcal{A} \cap \mathcal{B}) \oplus \mathcal{C}$ are given on Page 884.



7-3-27

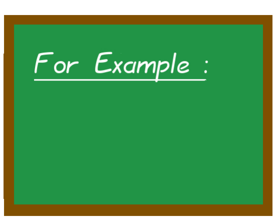
Suppose you are organizing the results from a marketing survey of ice-cream customers. Let $\mathcal{C}$ indicate those who preferred a simple ice-cream cone, as compared to something complicated like a banana split or sundae. Let $\mathcal{V}$ be someone who ordered vanilla, let $\mathcal{S}$ indicate strawberry, and let $\mathcal{P}$ indicate pistachio. What do the following sets indicate?

1. $\mathcal{V} \cap \mathcal{C}$?
2. $\mathcal{S}^c \cap \mathcal{C}$?
3. $\mathcal{P} \cap \mathcal{C}^c$?
4. $\mathcal{S} \cup \mathcal{P}$?
5. $\mathcal{C} \cup \mathcal{V}$?

The answers can be found on Page 885.

In the Summer of 2016, I made an extended visit to Bulgaria, and visited with various Bulgarian professors of mathematics, including the prestigious Technical University of Sophia. I learned a lot about the Bulgarian system of education, including the high schools, which differ tremendously from our own (in the USA). The Bulgarian system creates a lot of expertise in foreign languages (Russian, German, French, and English) as well as a lot of expertise in math.

In the next box, we'll analyze a hypothetical company importing luxury goods into Bulgaria. Among all the employees in the company, a good number will be knowledgable in French, Russian, or English, with smaller numbers speaking two out of those three, and very few employees being quadri-lingual, speaking those three along with their native Bulgarian.



7-3-28

Continuing with the previous box, we're going to imagine that we're creating a database of employees of a Bulgarian import firm. Queries will be made to the database to find employees with various arrangements of language skills. Our job is to translate those queries into set theory. Let $\mathcal{R}$ signify the employees who speak Russian, let $\mathcal{F}$ signify the employees who speak French, and let $\mathcal{E}$ signify the employees who speak English.

1. Employees who speak both Russian and English: $\mathcal{R} \cap \mathcal{E}$.
2. Employees who speak French or Russian: $\mathcal{F} \cup \mathcal{R}$.
3. Employees who speak all three: English, Russian and French: $\mathcal{E} \cap \mathcal{F} \cap \mathcal{R}$.
4. Employees who speak English or French but not both: $\mathcal{E} \oplus \mathcal{F}$.

We'll continue with some harder ones in the next box.

Continuing with the previous box, we have four more queries to translate from English into Set Theory. These are a bit harder.

5. Employees who speak Russian, but neither English nor French: This one can be written in two different ways:

$$\mathcal{R} \cap (\mathcal{E} \cup \mathcal{F})^c = \mathcal{R} \cap (\mathcal{E}^c \cap \mathcal{F}^c)$$

Note: We can leave the parentheses off and write: $\mathcal{R} \cap \mathcal{E}^c \cap \mathcal{F}^c$ because of reasons that were explained on Page 866.

6. Employees who speak French, and either English or Russian: This one can also be written in two different ways:

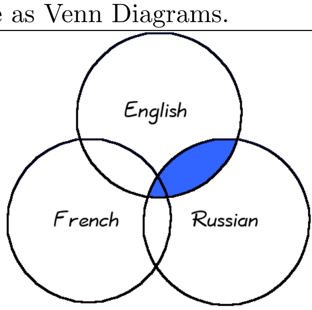
$$\mathcal{F} \cap (\mathcal{E} \cup \mathcal{R}) = (\mathcal{F} \cap \mathcal{E}) \cup (\mathcal{F} \cap \mathcal{R})$$

7. Employees who speak two or more languages: (This one is tough.)

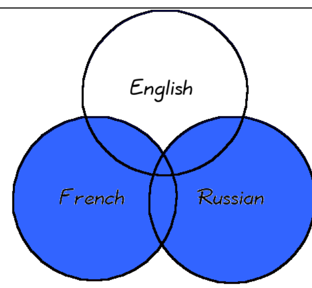
$$(\mathcal{F} \cap \mathcal{E}) \cup (\mathcal{R} \cap \mathcal{E}) \cup (\mathcal{F} \cap \mathcal{R})$$

8. Employees who speak Russian and exactly one other language of the three: $\mathcal{R} \cap (\mathcal{F} \oplus \mathcal{E})$.

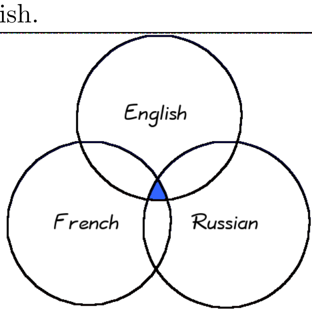
I would like to help you understand the example of the previous three boxes, by showing you what the answers look like as Venn Diagrams.



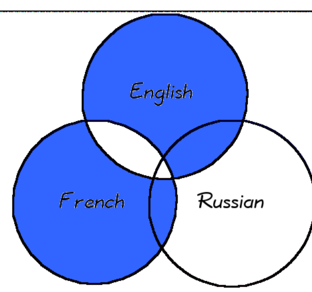
1. Employees who speak both Russian and English.



2. Employees who speak French or Russian.



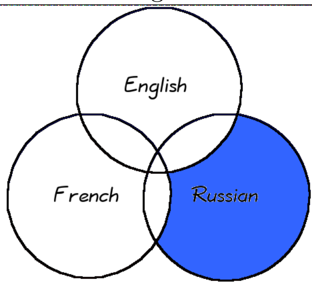
3. Employees who speak all three: English, Russian and French.



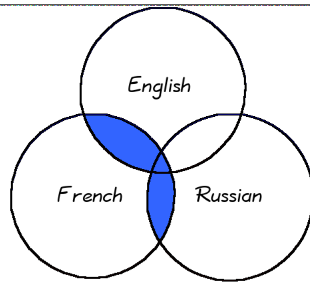
4. Employees who speak English or French but not both.

We will continue in the next box.

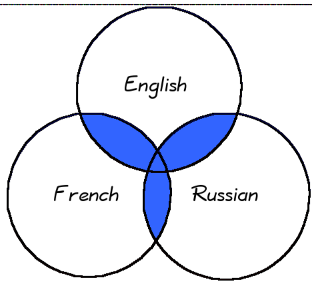
Here are the diagrams for the harder questions.



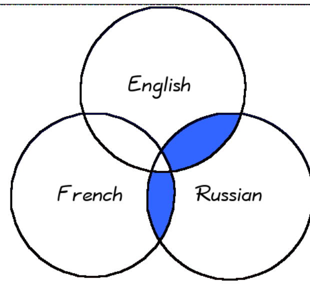
5. Employees who speak Russian, but neither English nor French.



6. Employees who speak French, and either English or Russian.

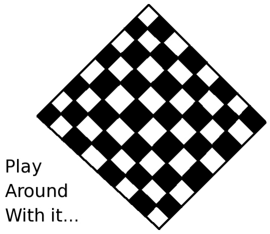


7. Employees who speak two or more languages.



8. Employees who speak Russian and exactly one other language of the three.

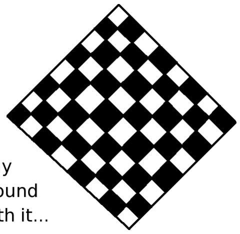
The next problem appeared in *Finite Mathematics & Its Applications* by Larry J. Goldstein, David I. Schneider and Martha J. Siegel. It was in Chapter 5, Section 1, Exercises 27–32, of the Eleventh Edition.



\# 7-3-29

An automobile insurance company classifies applicants by their driving records for the previous three years. Let $\mathcal{S}$ represent applicants who have received speeding tickets, $\mathcal{A}$ represent applicants who have caused accidents, and $\mathcal{D}$ represent applicants who have been arrested for driving while intoxicated. Represent the following queries in set-theory notation.

1. Applicants who have not received speeding tickets? [Answer: $\mathcal{S}^c$.]
2. Applicants who have caused accidents and been arrested for drunk driving? [Answer: $\mathcal{A} \cap \mathcal{D}$.]
3. Applicants who have received speeding tickets, caused accidents, or been arrested for drunk driving? [Answer: $\mathcal{S} \cup \mathcal{A} \cup \mathcal{D}$.]
4. Applicants who have not been arrested for drunk driving, but have either received speeding tickets or have caused accidents? [Answer: you can write "$\mathcal{D}^c \cap (\mathcal{S} \cup \mathcal{A})$" or you can write "$(\mathcal{D}^c \cap \mathcal{S}) \cup (\mathcal{D}^c \cap \mathcal{A})$."]
5. Applicants who have not both caused accidents and received speeding tickets? [Answer: you can write either "$(\mathcal{A} \cap \mathcal{S})^c$" or also "$(\mathcal{A}^c \cup \mathcal{S}^c)$."]



Play Around With it...

7-3-30

Referring to the previous box, here are some harder ones!

6. Applicants who have not both caused accidents and received speeding tickets, but who have been arrested for drunk driving? [Answer: you can write "$(\mathcal{A} \cap \mathcal{S})^c \cap \mathcal{D}$" or also "$(\mathcal{A}^c \cup \mathcal{S}^c) \cap \mathcal{D}$" as well as possibly "$(\mathcal{S}^c \cap \mathcal{D}) \cup (\mathcal{A}^c \cap \mathcal{D})$."]

7. Applicants who have [either] not caused accidents or who have not been arrested for drunk driving? [Answer: you can write "$\mathcal{A}^c \cup \mathcal{D}^c$" or also "$(\mathcal{A} \cap \mathcal{D})^c$."]

8. Applicants who have not been arrested for drunk driving, but who have either a speeding ticket or an accident, yet not both a speeding ticket and an accident. [Answer: this can be written $\mathcal{D}^c \cap (\mathcal{A} \oplus \mathcal{S})$ or also as $(\mathcal{D}^c \cap \mathcal{A}) \oplus (\mathcal{D}^c \cap \mathcal{S})$.]

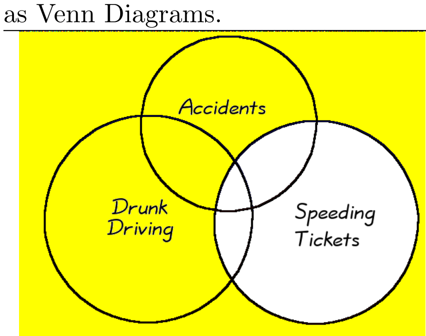
It was through using DeMorgan's Law of Sets, which we discussed on Page 862, that I was able to rapidly figure out the multiple different ways of answering some of the questions in the previous box. However, that need not concern us at this time.

One clear case of DeMorgan's Law is #7 from the previous box:

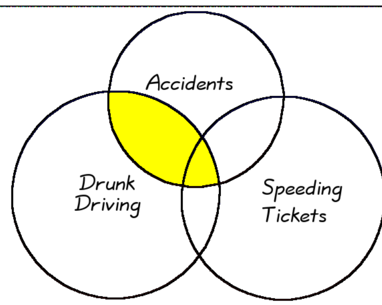
$$\mathcal{A}^c \cup \mathcal{D}^c \qquad \text{is equivalent to} \qquad (\mathcal{A} \cap \mathcal{D})^c$$

If any aspects of the previous long problem confused you, don't worry just yet. Let's re-examine it from the point of view of Venn Diagrams.

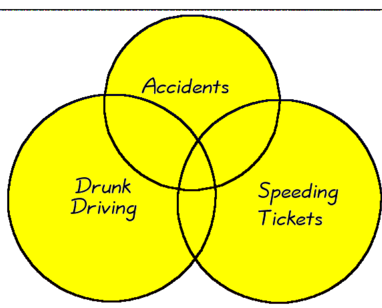
I would like to help you understand the previous two checkerboard boxes, by showing you what the answers look like as Venn Diagrams.



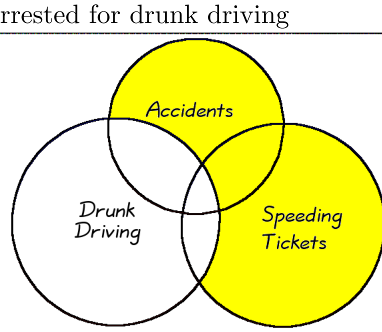
1. Applicants who have not received speeding tickets



2. Applicants who have caused accidents and been arrested for drunk driving



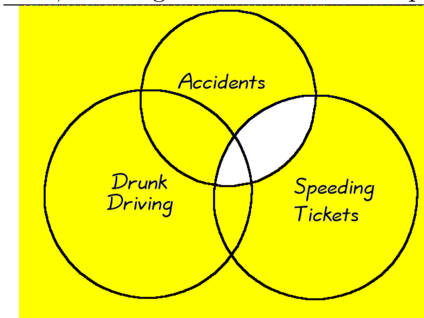
3. Applicants who have received speeding tickets, caused accidents, or been arrested for drunk driving



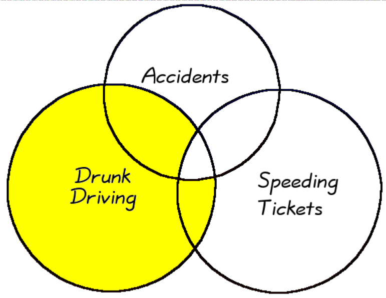
4. Applicants who have not been arrested for drunk driving, but have [either] received speeding tickets or [have] caused accidents

We will continue in the next box.

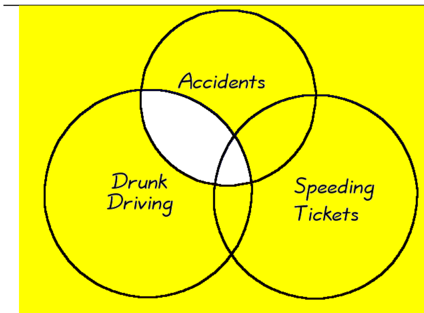
Now, the diagrams for the harder questions:



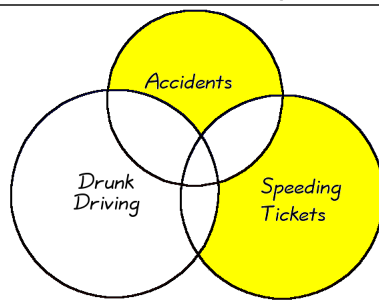
5. Applicants who have not both caused accidents and received speeding tickets



6. Applicants who have not both caused accidents and received speeding tickets, but who have been arrested for drunk driving



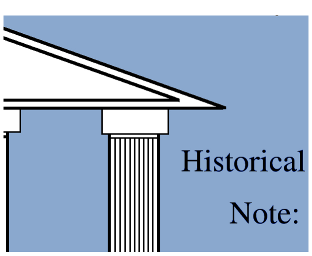
7. Applicants who have [either] not caused accidents or who have not been arrested for drunk driving



8. Applicants who have not been arrested for drunk driving, but who have either a speeding ticket or an accident, yet not both a speeding ticket and an accident.

As you can see by the previous example, you have to read English sentences *extremely carefully* when dealing with problems of set theory. Very subtle distinctions can turn on a single word, such as "and," "or," and "not."

Even the presence or absence of a comma can change the meaning of a sentence greatly, especially in matters of law. The most famous example is the case of "The Man who was Hanged by a Comma."



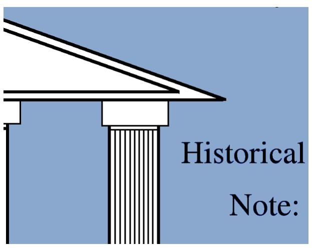
You might or might not have heard of the case of "The Man who was Hanged by a Comma." For anyone who is not an expert in punctuation and grammar (which you have surely noticed by now are weaknesses of mine), the fact that the punctuation of a law might determine whether or not someone gets the death sentence is a bit unsettling.

Because outlandish claims require excellent evidence, especially in the age of the internet when outlandish claims tend to spread rapidly, I have researched this point very carefully. In the next four boxes, I will tell this story.

During World War I (1914–1919) Britain (and several allied countries) were fighting Germany and its allies. At this time, Ireland was a part of the UK, and many in Ireland desired independence. The German government decided to foment an armed rebellion in Ireland, with the idea that a long-running and organized uprising would draw thousands of British troops away from the trenches, weakening Britain's military situation.

Roger David Casement, (1864–1916), was a British diplomat with an extremely successful career. He was knighted in 1911 for his work in investigating human-rights abuses in The Congo and in Peru. Because of his diplomatic experience, Casement was chosen to represent Ireland in negotiations with Count Johann Heinrich von Bernstorff (1862–1939), the German ambassador to the United States, with the idea that Germany would supply arms and soldiers for the rebellion. Casement then travelled to Germany to organize the details. At one point, the offer included 20,000 rifles and 10 machine guns, as well as ammunition.

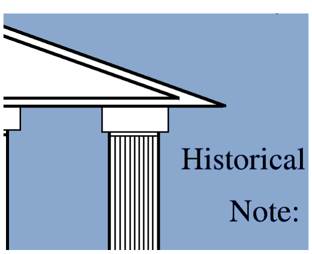
The story continues in the next box.



Continuing with the previous box, on April 21st, 1916, Casement landed on the west coast of Ireland at 3 AM. He travelled from Germany to Ireland in a U-Boat submarine, along with two members of the Irish Brigade. According to *Spies of the First World War*, by Bill Price,

> "The landing was observed by a number of people, despite happening at about 3 a.m., and while his two companions were not caught, Casement was picked up by the Irish police almost immediately."

The uprising went forward anyway, and is now called "The Easter Rising." The rebellion lasted only six days. The German weapons did not arrive in time, because the (British) Royal Navy intercepted the cargo ship carrying them.



Continuing with the previous box, during the uprising and afterward, Casement was held in Brixton Prison (near London) awaiting trial on charges for treason. In the British legal system, just like in the American legal system, a person must be charged with breaking a specific law. While common sense would dictate that Casement's actions must be illegal, it was very difficult to actually find a law that prohibited what he had done. An extensive search of British law had to take place. According to *Spies of the First World War*, by Bill Price,

> "This required a specific legal interpretation of the Treason Act of 1351, and would lead to Casement becoming known as 'the man who was hanged by a comma,' because it was said that the case for the defense was based on how the original document was punctuated. The fact that Casement had conspired with the enemy was clear, and he had admitted as much under interrogation, so once it had been deemed that his actions in Germany could be considered treasonous, a guilty verdict and sentence of death became inevitable."

As you can see, after debating the meaning and legal impact of the punctuation of the Treason Act of 1351, written in medieval Norman-French 565 years before Casement's arrest, he was found guilty and sentenced to death. Casement was hanged on August 3rd, 1916, precisely 104 days after he was arrested.

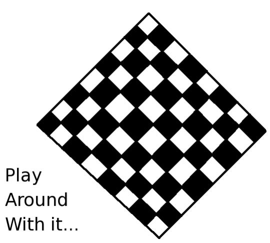
If you are interested in the fine point of law debated at Casement's trial, as described in the previous box, it can be summarized this way. (If that does not interest you, please feel free to skip to the next checkerboard box.) The phrase at issue was "in the realm or elsewhere." The prosecution claimed that this phrase indicated that the Treason Act applied whether or not the crimes were committed "in the realm or elsewhere." (All of Casement's actions were taken in New York, and in Germany—obviously neither New York nor Germany is part of Britain.)

The defense claimed that the law could only apply to actions on British soil, and not to actions taken overseas. The defense claimed that the phrase "in the realm or elsewhere" applied to identifying who could qualify as "the King's enemies." This is just like the phrase in US law, "all enemies, foreign or domestic" which appears in the oath of office, sworn by all members of the US military and all US civil servants, including myself when I worked for the NSA. If the defense could successfully argue that the Treason Act of 1351 applied only to actions taken on British soil, then Roger Casement could not be guilty.

From the above discussion, it's really clear that it was crucial to determine which parts of the key sentence were governed by the phrase "in the realm or elsewhere," and that is a matter of punctuation. This determination was complicated by the fact that the law was written in medieval Norman-French. (Because of all that happened in 1066, laws in England were not written in English until the 1400s, during The Hundred Years War.)

In summary, the court's conclusion was that the prosecution's grammatical interpretation was correct, and therefore Roger Casement was convicted and hanged.

The next problem appeared in *Finite Mathematics & Its Applications* by Larry J. Goldstein, David I. Schneider and Martha J. Siegel. It was in Chapter 5, Section 1, Exercises 21–26, of the Seventh Edition.



\# 7-3-31

Consider the divisions of a corporation, $\mathcal{D}$. They have been categorized by those that have made a profit $\mathcal{P}$, those that experienced increased revenue $\mathcal{R}$, and those that experienced increased labor costs $\mathcal{C}$. Below are some human-language sentences. Translate them into set theory. (The answers are given on Page 885.)

1. Divisions that had increases in labor costs or total revenue.
2. Divisions that did not make a profit.
3. Divisions that either had an increase in revenue or an increase in profit, but not both.
4. Divisions that made a profit yet experienced an increase in labor costs.
5. Divisions where the labor costs decreased or stayed the same.
6. Divisions that had an increase in labor costs and either were unprofitable or did not increase their total revenue.
7. Profitable divisions with increases both in total revenue and labor costs.
8. Divisions that either were unprofitable, did not increase in total revenue, or did not increase in labor costs.

Play
Around
With it...

\# 7-3-32

For any airline, frequent flyer programs are vitally important. When a customer is close to reaching the next status, they might use the airline even when a competing airline is offering a cheaper or more convenient flight. Let's imagine that you've got an internship working for a major airline. Your airline is trying to figure out who should have platinum status, gold status, or silver status, this year. Here are the sets you are working with.

- Let $\mathcal{S}$ signify customers who have flown more than three qualifying segments this year.
- Let $\mathcal{M}$ signify customers who have flown more than 20,000 miles this year.
- Let $\mathcal{G}$ signify customers who had gold status last year.

Now, convert the following queries into set theory notation. The answers are given on Page 887.

1. Customers who have not flown more than three qualifying segments this year.
2. Customers who have flown more than 20,000 miles this year and who had gold status last year.
3. Customers who have flown more than 20,000 miles this year or who have flown more than three qualifying segments, but not both.
4. Customers who either flew more than 20,000 miles this year, who have flown more than three qualifying segments this year, or who had gold status last year.

The problem continues in the next box.

Play
Around
With it...

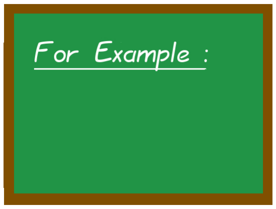
\# 7-3-33

Continuing with the previous box, convert the following queries into set theory notation.

5. Customers who did not have gold status last year, but who have either flown more than three qualifying segments this year or flown more than 20,000 miles this year.
6. Customers who have not both flown more than 20,000 miles this year and flown more than three qualifying segments this year.
7. Customers who have not both flown more than 20,000 miles this year and flown more than three qualifying segments this year, but who had gold status last year.
8. Customers who have not flown more than 20,000 miles this year or who did not have gold status last year.

The answers, including some Venn Diagrams to clarify the answers, can be found on Page 887.

Now, let us return to counting problems. Often it is difficult to survey people about who they will vote for, as many people are hesitant to talk about politics in public. It is both obvious and widely accepted that the people who are very willing to talk about politics do not form a representative subset of the population at large, but rather tend to over-count extremes and under-count the center (moderates). Let us imagine that one pollster, a friend of yours, decides to try getting people to reveal their views with long conversation, and he comes up with the following information.



7-3-34

Suppose there's an election coming up between a Democrat and a Republican, and a pollster tries to survey a large group of people. As it comes to pass, only 39 people were willing to talk to him. Of those, 14 said they would not be satisfied if the Democratic candidate won; contrastingly, 18 said they would not be satisfied if the Republican candidate won. Moreover, 6 people said they would not be satisfied with either candidate. We're now going to construct a Venn Diagram, with those who would be satisfied with the Democratic candidate denoted on the left and those who would be satisfied with the Republican candidate on the right.

First, we know 6 people won't be satisfied with either candidate and the survey contained 39 people, so let's note those down in the appropriate places. We will continue in the next box.

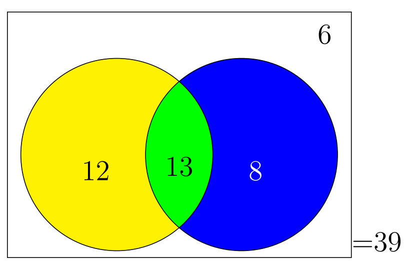
Continuing with the previous box, surely the left circle (including both the football-shaped part and the moon-shaped part), are people who would be satisfied with the Democratic candidate, thus the 14 who would not be satisfied must be located in the right moon-shaped area, or in the background. Since we have 6 in the background, we must have 8 in the right moon-shaped area.

Likewise, the right circle, including both the right moon-shaped region and the football-shaped region in the middle, are people who would be satisfied with the Republican candidate. Then the 18 people who said that they would not be satisfied must be either in the background, or the left moon-shaped region. Since there are 6 in the background, there must be 12 in the left moon-shaped region.

The diagram is now complete, except for the football-shaped center, which has

$$39 - 12 - 8 - 6 = 13$$

This is the Venn Diagram for the previous box.



It is useful to see that the diagram is consistent with the data.

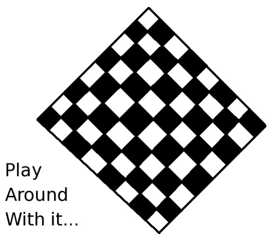
- We were told that 39 people were in the survey. $12 + 8 + 13 + 6 = 39$. Good.
- We were told that 14 would not be satisfied with the Democrat. $8 + 6 = 14$. Good.
- We were told that 18 would not be satisfied with the Republican. $12 + 6 = 18$. Good.
- We were told that 6 people would not be satisfied with either candidate. Well, we have 6 noted in the background, so that's good too.

Since everything matches up with the chart, we know that we have gotten the problem entirely correct.

You might be wondering what the previous problem has to do with our present discussion of complements. As it turns out, the numbers in the problem are highly related to complements. If $\mathcal{D}$ represents people who would be happy with the Democratic candidate (that's the left circle), and $\mathcal{R}$ represents people who would be happy with the Republican candidate (that's the right circle), then we could write:

- "14 said they would not be satisfied if the Democratic candidate won" as $\mathcal{D}^c$.
- "18 said they would not be satisfied if the Republican candidate won" as $\mathcal{R}^c$.
- "6 people said they would not be satisfied with either candidate" as $(\mathcal{D} \cup \mathcal{R})^c$.

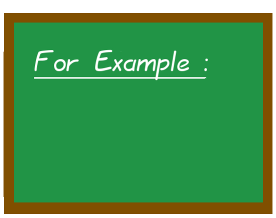
Perhaps this might make you imagine that maybe we could solve a Venn Diagram problem "inside out" by only working with the complements. The circles would be $\mathcal{D}^c$ and $\mathcal{R}^c$. In fact, this is sometimes useful, and you'll find examples of that later in this module.



7-3-35

The local news is going to be adding coverage of additional local sports teams. The two new sports might be water polo and soccer. They survey 120 viewers, and 35 said they would not watch soccer; 30 said they would not watch water polo. Among those, 24 said they would watch neither sport. Draw a Venn diagram to represent this situation, where the left circle represents people who will watch soccer, and the right circle represents people who will watch water polo. Then answer how many viewers can be expected to watch both sports.

The answer can be found on Page 889.



7-3-36

We are ready to see how introducing the variable x can help solve a counting problem. Suppose that two competing news stations, DMM and FerretNews, are being considered as possible venues for advertising a product that your company manufactures. After performing a marketing survey, you learn that out of 2000 people, 1251 watch DMM, and 751 watch FerretNews. A previous survey found out that if your company advertises on both shows, 1400 viewers out of 2000 will be reached. Assuming that is still true, how many people watch both programs?

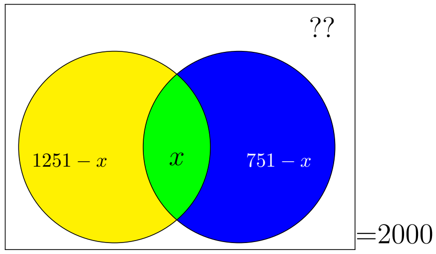
If we let $\mathcal{D}$ be DMM viewers and $\mathcal{F}$ be FerretNews viewers, we are obviously being asked for $\#(\mathcal{D} \cap \mathcal{F})$ but we are given $\#\mathcal{D}$ as well as $\#\mathcal{F}$, and $\#(\mathcal{D} \cup \mathcal{F})$. Each of our data points, namely 2000, 1251, 751, and 1400, refers to a set that is multiple regions of the Venn Diagram. None of those data points represents a single region (alone) of the Venn Diagram. It looks as though we have nothing to write down as a starting point. Are we trapped? Is there no way out? What shall we do?!

It turns out that by using an x for one of the regions that we do not know, we can solve the problem. So let us "go for the center" and put an x in the football-shaped intersection region. Generally, I do not add a variable to a Venn Diagram question unless I must, and if I do add one, I try to go for the most central as-yet-unknown region.

Clearly the moon-shaped region for $\mathcal{D}$ alone will be $1251 - x$ and the moon-shaped region for $\mathcal{F}$ alone will be $751 - x$.

Now we have the Venn Diagram as shown in the next box.

Here is the Venn Diagram from the previous box, where I am placing $\mathcal{D}$ on the left and $\mathcal{F}$ on the right:



We know the union is clearly the sum of both moon-shaped regions and the football-shaped region. That sum gives the equation

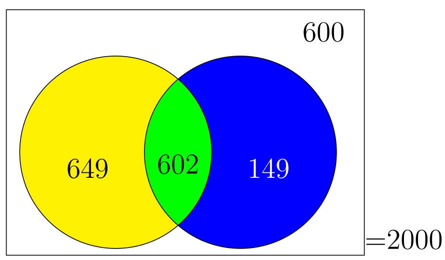
$$1400 = (1251 - x) + x + (751 - x)$$

which is equivalent to

$$1400 = 2002 - x$$

thus clearly $x = 2002 - 1400 = 602$. This is what we were asked to find, so we stop now.

However, either curiosity, or a desire to check your work, might drive you to desire to find the complete final Venn Diagram of the previous example, with all the numbers filled in. After a little movement of your pencil, you will discover that to be

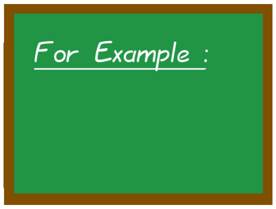


Now we must check to see if that tabulation is consistent with the given data. We will do that in the next box.

Here we are checking our work for the previous example.

- The circle for $\mathcal{D}$, on the left, indeed has $649 + 602 = 1251$ people. Good.
- Similarly the circle for $\mathcal{F}$, on the right, indeed has $602 + 149 = 751$ people. Good.
- Next, the union of the two circles is $649 + 602 + 149 = 1400$. Good.
- Finally, $649 + 149 + 602 + 600 = 2000$, and we were told there are 2000 people in the survey. Good.

Since everything matches up, we can be confident that we have gotten the question perfectly right.



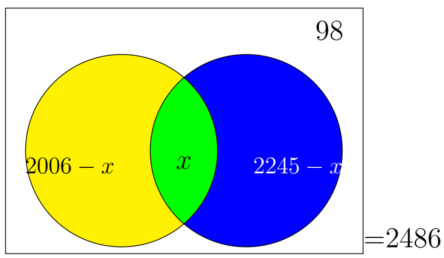
7-3-37

Suppose that college freshmen at a particular university commonly take both ENGL-101: *Freshman Composition* and HIST-111: *US History* in their first semester, unless they place out via "advanced placement credit," also called "AP-credit." Suppose the freshman class has 2486 students in it, and 480 opted to use AP-credit to escape the freshman composition course, and 241 managed to escape the history course. Suppose further that 98 people placed out of both. How many freshman are on the roster for the composition course? and for the history course?

First, we'll let $\mathcal{F}$ be those students registered for freshmen composition (left circle of the Venn Diagram), and $\mathcal{H}$ those registered for history (right circle of the Venn Diagram). Second, we observe that 98 people placed out of both courses, so that goes in the background, outside the two circles. This means that the two circles have $2486 - 98 = 2388$ students among them.

Since 241 people escaped history, then $2486 - 241 = 2245$ are registered for it; likewise, since 480 escaped freshman composition, then $2486 - 480 = 2006$ are registered for it. This is what we were asked, so we could stop here.

Now out of sheer curiosity, let's keep going, to complete the Venn Diagram and thus be better enabled to check our work. Let's "go for the center" again, and place an x there. Then we have freshmen composition on the left, so $2006 - x$ people are in the moon-shaped part (taking composition but not history) and x people go in the football-shaped part (taking both). Next, we have $2245 - x$ in the moon-shaped part (taking history but not composition), and then we have



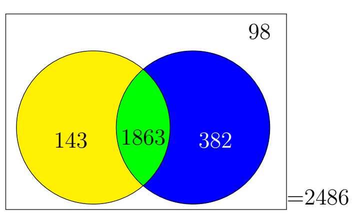
The key to finishing off the example of the previous two boxes is to realize that we know the union to be 2388. Therefore, we can write

$$2388 = (2006 - x) + (x) + (2245 - x)$$

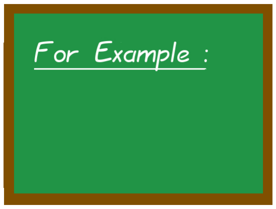
and that can be simplified to

$$2388 = 4251 - x$$

so we have $x = 1863$. Thus the number of people taking freshman composition only is $2006 - 1863 = 143$ and the number of people taking history only is $2245 - 1863 = 382$. The final Venn Diagram is



Note, it is very important to always make sure that you answer the question that was asked. In addition to constructing the Venn Diagrams, we were asked how many are registered for ENGL-101: *Freshman Composition*, to which we answer $143 + 1863 = 2006$ and how many are registered for HIST-111: *US History*, which comes to $382 + 1863 = 2245$.



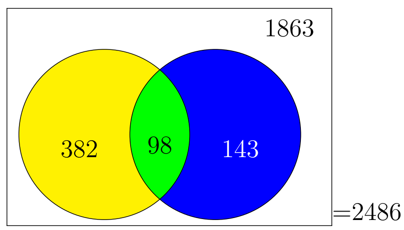
7-3-38

There is an alternative (and rather clever) way to approach the previous example. We simply let each circle represent the people who have placed out of something. Thus the left circle will be those who have placed out of writing, and the right circle will be those who have placed out of history. We have in the center, those who managed to escape both classes, or 98 students. Of course, that is the football-shaped region between the two circles.

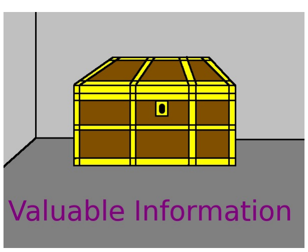
Then since 480 got out of the writing course in total, then $480 - 98 = 382$ only got out of freshman composition, and belong in the left moon-shaped part. Likewise, since 241 escaped history, then $241 - 98 = 143$ escaped history alone, and are taking freshman composition—they belong in the right moon-shaped part. Since the total freshman class is 2486, and we've accounted for $382 + 98 + 143 = 623$ people, then the background should have $2486 - 623 = 1863$ people.

We will continue in the next box.

Continuing with the previous box, notice how rapidly we have advanced to the Venn Diagram, which is

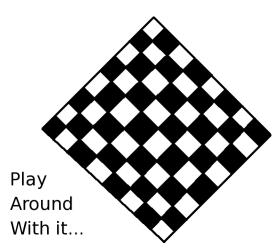


However, we have not yet answered the question that was asked! First, we must calculate how many students are registered for ENGL-101: *Freshman Composition*? The very people who have not placed out of it, which would be everyone outside the left circle, namely $143 + 1863 = 2006$. Likewise, how many are in HIST-111: *US History*? The people who have not placed out of it, or those outside the right circle, and that is $382 + 1863 = 2245$ students.



In the previous box, we saw that it was easier to calculate using the complements of what we want to know, rather than to calculate what we want to know.

This will be a repeated occasional theme throughout this chapter. Sometimes, by looking at the complement of a set, an event, or some property, you can easier calculate something that you are asked about. We will see this idea—the use of complements—again and again. It is kind of like turning a problem inside-out.



7-3-39

Let us consider that a sports arena is surveying frequent visitors to see what type of concession stand to offer. They survey 408 people and it turns out that 198 would like fried chicken, and 205 would like health food. However, 106 request both. Draw a Venn Diagram to encapsulate the situation and tell me how many people would like neither option.

(Place fried chicken on the left and health food on the right.) The answer can be found on Page 889. By the way, we will revisit this question again, on Page 901.

Play Around With it...

\# 7-3-40

A restaurant is contemplating a liquor license and the owner's cousin is asked to survey people as they are waiting to be seated. He asks if they like beer or wine. The cousin does write down that 104 customers like beer and 67 customers like wine. However, he did not write down how many people like both. He surveyed 135 people. All is not lost, however, as he remembers that only two couples (4 people) said that they neither liked wine nor beer. Draw a Venn Diagram to encapsulate the situation and tell me how many people like both wine and beer.

(Put wine drinkers on the left and put beer drinkers on the right.) The answer can be found on Page 889. By the way, we will revisit this question again, on Page 901.

DANGER !!!

Many times, I have seen students gleefully solve for every possible detail in a question, and make a very complete Venn Diagram. After all, that is kind of fun, especially compared to the other things on a math test. On the other hand, they might forget to answer a question that was asked.

In the last two boxes, many students would miss the inquiry "and tell me how many people would like neither option" as well as the inquiry "and tell me how many people like both wine and beer."

You always must be sure that you explicitly answer the question that was asked.

Play Around With it...

\# 7-3-41

A graduate student has taken a large number of statistics courses, and he sets out on an important survey. He's measuring what sorts of mass transit the typical commuter at his college uses. Being an experienced student, he is certain to note not only how many people are in each set, but the union and intersection as well.

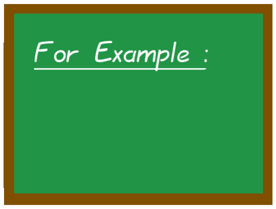
There were 207 people who said they used the subway, and 111 people who said that they used the bus. Furthermore, 82 people use both. Another 22 people said that they used neither, presumably walking or driving. However, our graduate student forgot to record how many people he spoke to!

Can that count be recovered from this data? Draw a Venn Diagram to encapsulate this situation and tell me how many people he interviewed. (Place the subway riders in the left circle and the bus riders in the right circle.) The answer can be found on Page 889.

Play Around With it...

\# 7-3-42

A survey of 542 supermarket shoppers is taken to analyze which fruits they buy regularly, in order to determine which ones should be put on sale. In the survey, 367 people regularly purchase bananas, and 218 regularly purchase apples. Then 59 people purchase neither. Construct a Venn Diagram of this situation, and then answer: how many people regularly buy either fruit? (Put bananas on the left and apples on the right in the Venn Diagram.) [Answer: to be found on Page 890.]



7-3-43

Your company is hiring 100 freshman interns. The requirements for the internship are a working knowledge of Microsoft Excel and Microsoft Word. However, an over-zealous recruiter has hired many students who are lacking in these skills. In particular, 23 of the 100 do not meet the criteria. Further examination of the intern pool reveals that 17 people do not know Excel, and 11 do not know Word. After some reflection, you realize that an intern who has one of the two skills is still useful, so you're only going to fire those people who are missing both skills. How many interns will you fire?

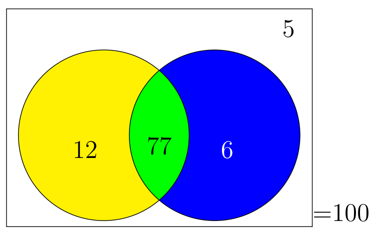
This is going to be one of those cases that is much easier to do if you let the sets be people who are missing a skill, rather than those that have a skill. But in order to demonstrate that, let's try it the long way first. Let's begin by drawing a Venn Diagram, and let $\mathcal{W}$ be those who know Microsoft Word (on the left), and let $\mathcal{E}$ be those who know Microsoft Excel (on the right).

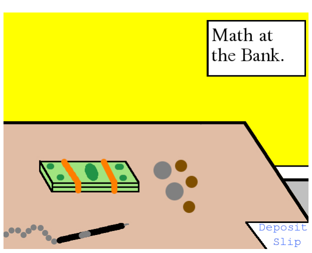
We will continue in the next box.

As we see from the previous box, since 23 people do not meet the criteria, then $100 - 23 = 77$ do meet the criteria, and they belong in the football-shaped part in the middle. We know that 17 people lack Excel skills, and 83 people have Excel skills. Then $83 - 77 = 6$ people have Excel skills but not Word skills.

On the other hand, 11 people lack Word skills, and so $100 - 11 = 89$ have Word skills, and this means that $89 - 77 = 12$ people have Word skills but not Excel skills.

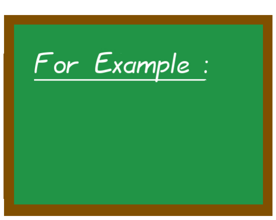
Last but not least, the people with either skill total to be $12 + 77 + 6 = 95$ interns. For this reason, you know that you will be firing $100 - 95 = 5$ people. The final Venn Diagram is below:





Before we continue, I can't help but share the following observation. Often the "moon-shaped" regions of a Venn Diagram indicate opportunities for growth. We saw this before, on Page 846.

For now, observe that if the company were to have an in-house crash-course in MS-Excel, then they could increase by 12 the number of interns who have both skills. Perhaps such a course can be found online for a relatively modest fee. There are many websites which offer short-courses as a series of videos on the web, and therefore, this might be a profitable move. However, an in-house crash-course on MS-Word would only gain them 6 interns, and is therefore less attractive.



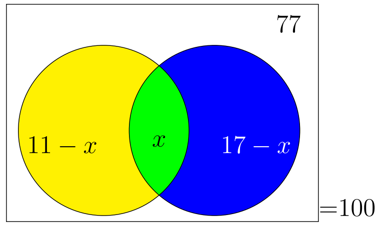
7-3-44

Now let's redo the previous example but instead, track the lack of skills. Let the left-circle be $\mathcal{W}^c$ and the right-circle be $\mathcal{E}^c$.

The 77 interns who meet the criteria lack nothing, so they go in the background. Then we have $100 - 77 = 23$ students who do not meet the criteria, to divide among our three circles. Therefore, let's "go for the center," and put an x in the middle, which represents students who lack both skills.

Then we have $17 - x$ in the "only Excel is missing" part, in the moon-shaped region on the right, and $11 - x$ in the "only Word is missing" part, in the moon-shaped region on the left. The Venn Diagram is in the next box.

Continuing with the previous box, we now have the following Venn Diagram.



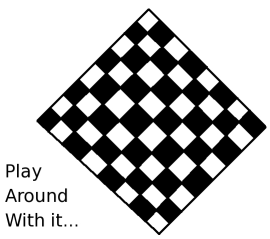
Adding up the three middle regions we have

$$23 = (11 - x) + x + (17 - x)$$

and this simplifies to

$$23 = 28 - x$$

and so $x = 5$. Thus you will fire 5 people, those who lack both skills.

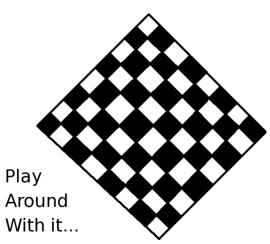


7-3-45

Let us imagine that you are interning with a fast-food company, and they are doing a safety audit. It turns out that 175 locations are in violation of some easily-remedied aspects of the building code, pertaining to having fire extinguishers and automatic defibrillators. In particular, 122 locations lack automatic defibrillators and 88 locations lack fire extinguishers. This isn't bad, considering that there are 895 locations total. Make a Venn Diagram that encapsulates this data. In particular, solve this problem two ways:

- Let the left circle be those stores that *have* defibrillators, and the right circle be those stores that *have* fire extinguishers.
- Let the left circle be those stores that *lack* defibrillators, and the right circle be those stores that *lack* fire extinguishers.

The answer is given on Page 890.



7-3-46

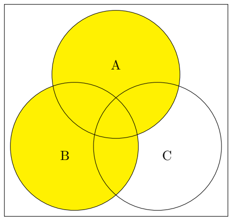
Perhaps you are interning with a staffing firm that specializes in hiring programmers. A software company is looking to expand, and your firm is hired to help them find the necessary talent. The programmers must know the computer languages `C++` and `Python`. Your firm receives 87 applications. Luckily, 75 applications include knowledge of at least one of the languages. As it comes to pass, 48 applicants know `Python`, which is a good start, but 31 applicants do not know `C++`. Make a Venn Diagram that encapsulates this data. In particular, solve this problem two ways:

- Let the left circle be those applicants who know `C++`, and let the right circle be those applicants who know `Python`.
- Let the left circle be those applicants who do not know `C++`, and let the right circle be those applicants who do not know `Python`.
- In both cases, further identify how many applicants know both languages.

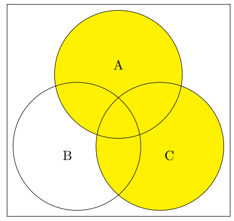
The answer will be given on Page 891. We will revisit this question again on Page 901.

You have now completed this module. All that remains is a listing of the answers to a few checkerboards from earlier in the module.

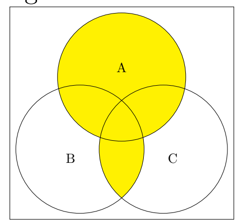
This is the answer to the checkerboard box on Page 859:



1
$\mathcal{A} \cup \mathcal{B}$

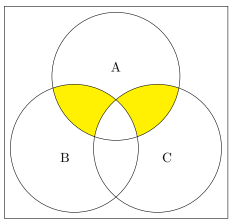


2
$\mathcal{A} \cup \mathcal{C}$

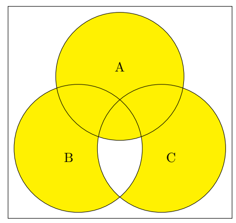


3
$(\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$

This is the answer to the checkerboard box on Page 859:



1
$\mathcal{A} \cap (\mathcal{B} \oplus \mathcal{C})$

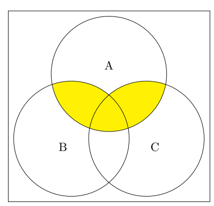


2
$\mathcal{A} \cup (\mathcal{B} \oplus \mathcal{C})$

From a checkerboard box on Page 865, the diagram below represents either side of the set equation:

$$\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$$

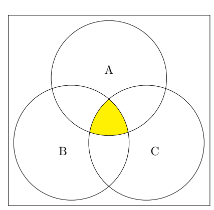
Here is the diagram:



From a checkerboard box on Page 865, the diagram below represents either side of the set equation:

$$\mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C}$$

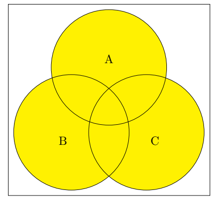
Here is the diagram:



From a checkerboard box on Page 865, the diagram below represents either side of the set equation:

$$\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}$$

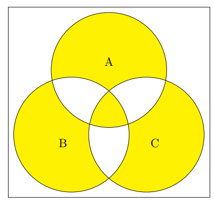
Here is the diagram:



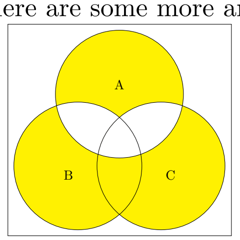
From a checkerboard box on Page 865, the diagram below represents either side of the set equation:

$$\mathcal{A} \oplus (\mathcal{B} \oplus \mathcal{C}) = (\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{C}$$

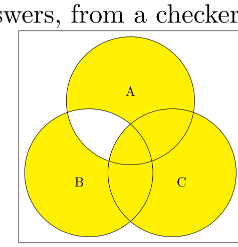
Here is the diagram:



Here are some more answers, from a checkerboard box on Page 866.

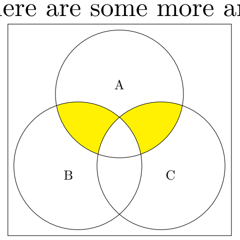


\# 1
$\mathcal{A} \oplus (\mathcal{B} \cup \mathcal{C})$

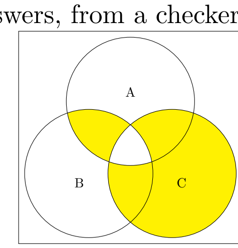


\# 2
$(\mathcal{A} \oplus \mathcal{B}) \cup \mathcal{C}$

Here are some more answers, from a checkerboard box on Page 866.



\# 1
$\mathcal{A} \cap (\mathcal{B} \oplus \mathcal{C})$



\# 2
$(\mathcal{A} \cap \mathcal{B}) \oplus \mathcal{C}$

Here are the answers to the ice-cream related question from Page 867.

- #1 is a simple vanilla cone;
- #2 is a simple cone with anything other than strawberry;
- #3 is a pistachio dessert that is not a simple cone;
- #4 is anything made up of pistachio or strawberry;
- #5 is any simple cone or any dessert with vanilla.

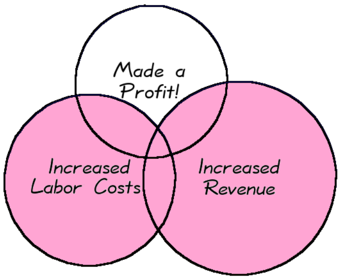
(Note, you should be looking for answers that are logically equivalent, not word-for-word the same.)

Here are the answers to the question about corporate divisions and set theory, from Page 873.

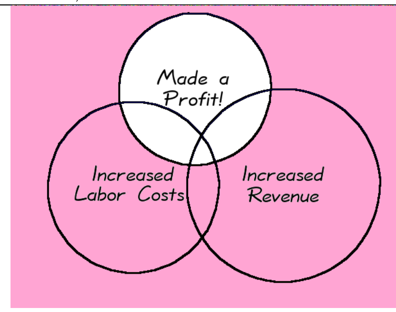
1. Divisions that had increases in labor costs or total revenue. [Answer: $\mathcal{C} \cup \mathcal{R}$.]
2. Divisions that did not make a profit. [Answer: $\mathcal{P}^c$.]
3. Divisions that either had an increase in revenue or an increase in profit, but not both. [Answer: $\mathcal{P} \oplus \mathcal{R}$.]
4. Divisions that made a profit yet experienced an increase in labor costs. [Answer: $\mathcal{P} \cap \mathcal{C}$.]
5. Divisions where the labor costs decreased or stayed the same. [Answer: $\mathcal{C}^c$.]
6. Divisions that had an increase in labor costs and either were unprofitable or did not increase their total revenue. [Answer: $\mathcal{C} \cap (\mathcal{P}^c \cup \mathcal{R}^c)$. Alternatively, $\mathcal{C} \cap (\mathcal{P} \cap \mathcal{R})^c$ or $(\mathcal{C} \cap \mathcal{P}^c) \cup (\mathcal{C} \cap \mathcal{R}^c)$.]
7. Profitable divisions with increases both in total revenue and labor costs. [Answer: $\mathcal{P} \cap \mathcal{R} \cap \mathcal{C}$.]
8. Divisions that either were unprofitable, did not increase in total revenue, or did not increase in labor costs. [Answer: $\mathcal{P}^c \cup \mathcal{R}^c \cup \mathcal{C}^c$. Alternatively, $(\mathcal{P} \cap \mathcal{R} \cap \mathcal{C})^c$.]

Keep in mind that when I show you alternative answers to a question, that you should be happy to find just one of the available choices. It would be extremely unusual to ask someone to find *all possible* ways of writing an English sentence in the language of set theory.

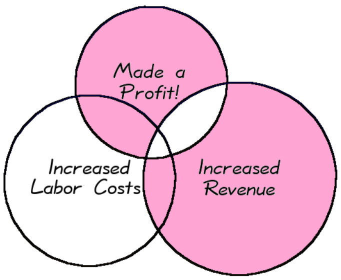
To help you understand the answers from the previous box, I have drawn some Venn Diagrams for you.



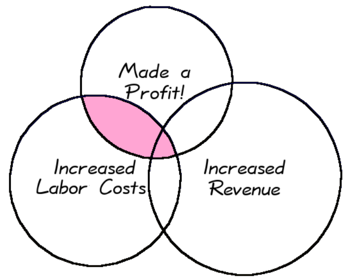
1. Divisions that had increases in labor costs or total revenue



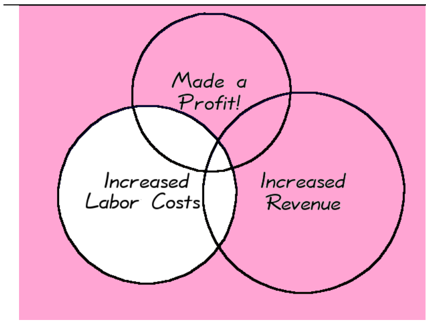
2. Divisions that did not make a profit



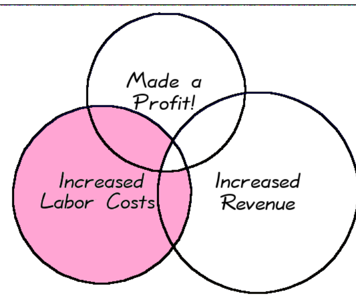
3. Divisions that either had an increase in revenue or an increase in profit, but not both



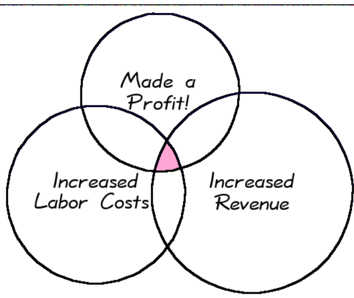
4. Divisions that made a profit yet experienced an increase in labor costs



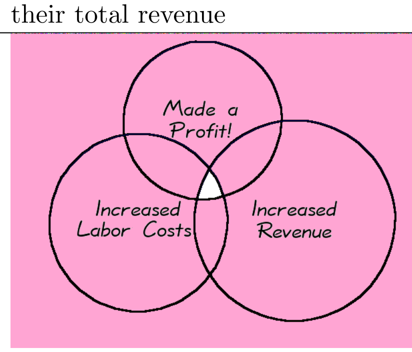
5. Divisions where the labor costs decreased or stayed the same



6. Divisions that had an increase in labor costs and either were unprofitable or did not increase their total revenue



7. Profitable divisions with increases both in total revenue and labor costs



8. Divisions that either were unprofitable, did not increase in total revenue, or did not increase in labor costs

Here are the answers to the question about frequent flyer programs and set theory.

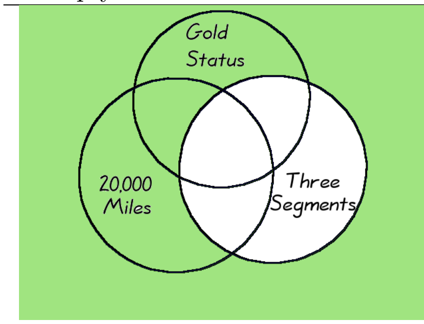
1. Customers who have not flown more than three qualifying segments this year. [Answer: $\mathcal{S}^c$.]
2. Customers who have flown more than 20,000 miles this year and who had gold status last year. [Answer: $\mathcal{M} \cap \mathcal{G}$.]
3. Customers who have flown more than 20,000 miles this year or who have flown more than three qualifying segments, but not both. [Answer: $\mathcal{M} \oplus \mathcal{S}$.]
4. Customers who either flew more than 20,000 miles this year, who have flown more than three qualifying segments this year, or who had gold status last year. [Answer: $\mathcal{M} \cup \mathcal{S} \cup \mathcal{G}$.]
5. Customers who did not have gold status last year, but who have either flown more than three qualifying segments this year or flown more than 20,000 miles this year. [Answer: $\mathcal{G}^c \cap (\mathcal{S} \cup \mathcal{M})$. Alternatively, you could write $(\mathcal{G}^c \cap \mathcal{S}) \cup (\mathcal{G}^c \cap \mathcal{M})$.]

We will continue in the next box.

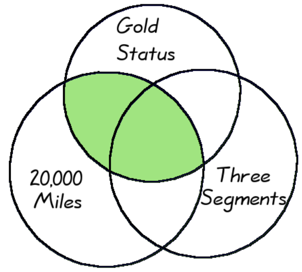
Continuing from the previous box…

6. Customers who have not both flown more than 20,000 miles this year and flown more than three qualifying segments this year. [Answer: $(\mathcal{M} \cap \mathcal{S})^c$. Alternatively, you could write $\mathcal{M}^c \cup \mathcal{S}^c$.]
7. Customers who have not both flown more than 20,000 miles this year and flown more than three qualifying segments this year, but who had gold status last year. [Answer: $(\mathcal{M} \cap \mathcal{S})^c \cap \mathcal{G}$. Alternatively, you could write $(\mathcal{M}^c \cup \mathcal{S}^c) \cap \mathcal{G}$ or $(\mathcal{M}^c \cap \mathcal{G}) \cup (\mathcal{S}^c \cap \mathcal{G})$.]
8. Customers who have not flown more than 20,000 miles this year or who did not have gold status last year. [Answer: $\mathcal{M}^c \cup \mathcal{G}^c$. Alternatively, you could write $(\mathcal{M} \cap \mathcal{G})^c$.]

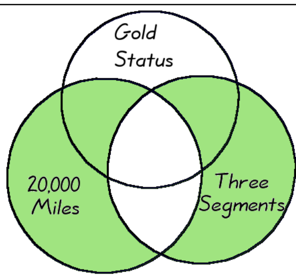
To help you understand the answers from the previous box, I have again drawn some Venn Diagrams for you.



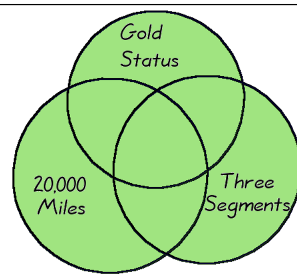
Customers who have not flown more than three qualifying segments this year



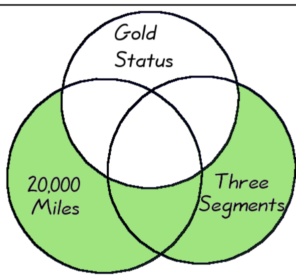
Customers who have flown more than 20,000 miles this year and who had gold status last year



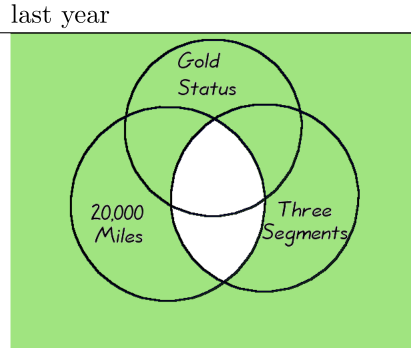
Customers who have flown more than 20,000 miles this year or who have flown more than three qualifying segments, but not both



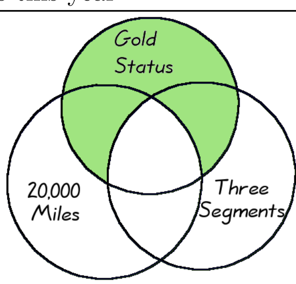
Customers who either flew more than 20,000 miles this year, who have flown more than three qualifying segments this year, or who had gold status last year



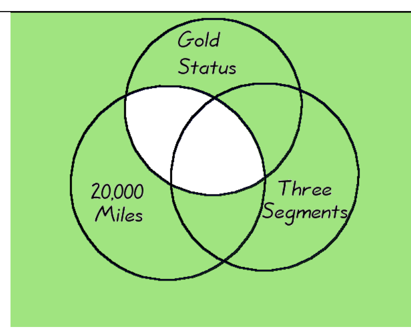
Customers who did not have gold status last year, but who have either flown more than three qualifying segments this year or flown more than 20,000 miles this year



Customers who have not both flown more than 20,000 miles this year and flown more than three qualifying segments this year

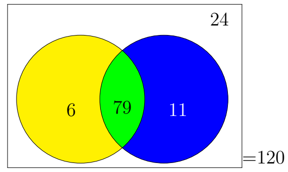


Customers who have not both flown more than 20,000 miles this year and flown more than three qualifying segments this year, but who had gold status last year

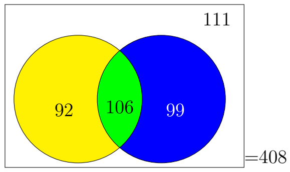


Customers who have not flown more than 20,000 miles this year or who did not have gold status last year

This is the solution to the soccer & water polo problem from Page 876.

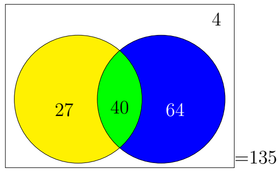


Here is the solution to the question about fried chicken and health food in a stadium (from Page 879).



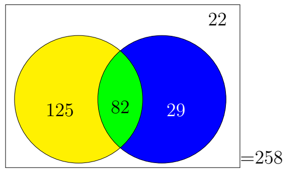
There was also an ancillary question: how many people like neither option? There are actually 111 people who would like neither option, so maybe a third concession should be thought of.

Here is the solution to the question about wine and beer on Page 880.



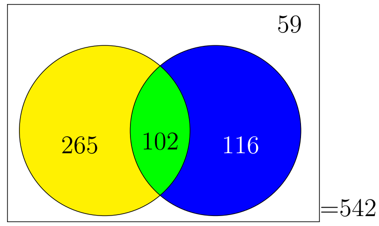
There was also an ancillary question: how many people like both wine and beer? The number of people who like both wine and beer is 40.

Here is the solution to the question about mass transit on Page 880.



There was also an ancillary question: how many people were interviewed? The graduate student interviewed 258 people.

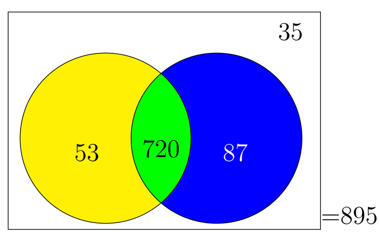
Here is the solution to the question about apples and bananas.



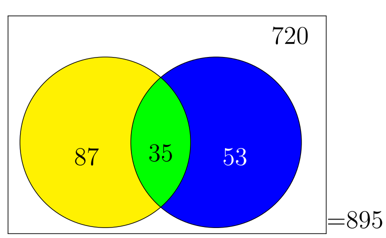
There also is the ancillary question: 483 shoppers regularly buy either bananas or apples.

This is the solution to the problem about the fast-food company, automatic defibrillators and fire extinguishers, from Page 882.

- If we let the left circle be those stores that have defibrillators, and the right circle be those stores that fire extinguishers, we obtain:

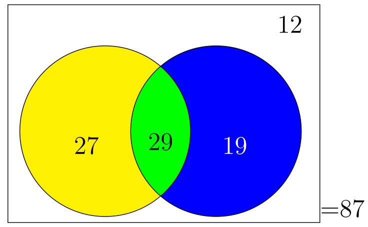


- If we let the left circle be those stores that lack defibrillators, and the right circle be those stores that lack fire extinguishers.

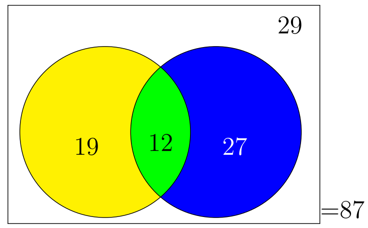


Here are the answers to the question about the staffing firm looking for `C++` and `Python` programmers, from Page 882.

- If we let the left circle be those applicants who know `C++`, and let the right circle be those applicants who know `Python`, then we obtain



- If we let the left circle be those applicants who do not know `C++`, and let the right circle be those applicants who do not know `Python`, then we obtain



In either case, the number of applicants who know both languages is 29.