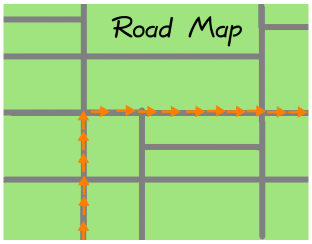
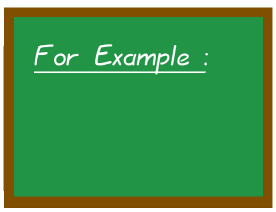
Module 7.9: The Combinations Principle and the Handshake Principle



The combinations principle is the most important of the six principles, and it will be the most frequently used for the remainder of this chapter. We can use it to solve a wide array of problems, including from subjects as diverse as inventory planning, reliability, and dispute resolution. There is also a special case of the combinations principle, called the handshake principle.

In order to properly explain this principle, I'm going to begin with two easy examples, but we will move on to harder problems soon enough.



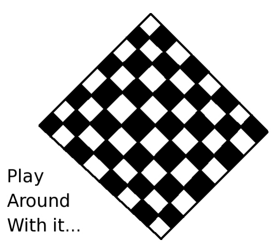
7-9-1

Our opening example is a bit bizarre, but it will make the entire module much easier to understand, so please bare with me. Suppose there is a group of secret agents that need to have code names to protect their identity. Perhaps each secret agent will have a codeword consisting of three letters taken from the set

$$\{A, B, C, D, E\}$$

Further suppose that we place two restrictions on the codewords. The first restriction is that no letter can be repeated. The second is that we want no agent to be given a codeword that is a reordering of the codeword of another agent. (This could have many advantages, especially if messages are being written/typed under stressful situations; if you are worried about counter-intelligence agents arresting you, surely you could accidentally interchange two letters when writing a secret message.) How many possible codewords (and therefore, secret agents), can there be?

The first restriction is one that we already know how to handle, but the second one is new. We will solve this problem over the next few boxes.



7-9-2

Using the principles of combinatorics that you already know, answer the following two questions.

- How many three-letter codewords can be formed using the letters "A" through "E," without restrictions?
  [Answer: $5^3 = 125$.]
- How many three-letter codewords can be formed using the letters "A" through "E", with the restriction that no letter can be repeated?
  [Answer: $P_{5,3} = (5)(4)(3) = 60$.]

Considering the problem of the last two boxes, as you've probably guessed, the combinations principle will solve the problem when both restrictions are present. Namely, no letter can be repeated, and no agent can be given a codeword that is a reordering of the codeword of another agent. On the other hand, we don't know the combinations principle just yet. With that in mind, let's make a list.

We will do that in the next box.

Continuing from the previous few boxes, let's make a list of all codewords formed using the letters "A" through "E", with the first restriction—no letter can be repeated. Then we will see how to resolve the challenge of the second restriction. Here is my list of codewords:

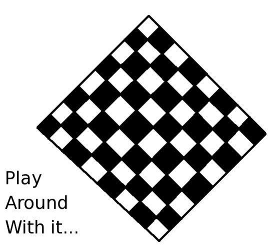
ABC, ACB, BCA, BAC, CAB, CBA,
ABD, ADB, BAD, BDA, DBA, DAB,
ABE, AEB, BAE, BEA, EAB, EBA,
ACD, ADC, CAD, CDA, DCA, DAC,
ACE, AEC, CAE, CEA, EAC, ECA,
ADE, AED, DEA, DAE, EAD, EDA,
BCD, BDC, CBD, CDB, DBC, DCB,
BCE, BEC, CEB, CBE, EBC, ECB,
BDE, BED, DEB, DBE, EBD, EDB,
CDE, CED, DCE, DEC, ECD, EDC.

Please don't be scared by the size of that list. We will solve the problems of this module mathematically, not by making lists. We will check this list in the next box.

We should take a moment to double-check my list in the previous box. First, take a moment to see that every possible combination is actually there. That might be hard to tell. One trick is to verify that no 3-letter codeword appears in the table more than once, and that every entry in the table is a valid codeword. If that's the case, and if we've written the correct number of codewords ($P_{5,3} = 60$) then we know we have a complete and correct list. It is easy to see that we have 6 entries in each row, and 10 rows. Thus there are $(6)(10) = 60$ entries. You might like to look through the list to be sure I have not duplicated a codeword.

If you look closely at that table of codewords, you'll see that each row consists of six codewords that are actually re-orderings of each other. The codewords in each row have the same set of letters (in terms of set theory), but they each show one of the six different possible orderings. Using the factorial principle, we know there are 3! ways to order 3 objects, so we expect to see $3! = 6$ codewords in each row—and we do see six codewords in each row!

Because each row has 6 codewords, and there are 60 codewords in the table, there must be $60/6 = 10$ rows. Thus we can issue ten codewords at most, by taking exactly one codeword from each row.



7-9-3

Let's make a variation on the previous long sequence of boxes. Suppose instead that we had an alphabet of the first nine letters ("A" through "I"), and that we chose four letter codewords.

- How many codewords can I make with that 9-letter alphabet, knowing that I must not repeat any letters inside of a codeword? [Answer: $P_{9,4} = 3024$.]

Note: This means my table will have 3024 entries in it.

- How many ways can a codeword that is four letters long be reordered? [Answer: $4! = 24$.]

Note: This means that each row will have 24 entries in it.

- How many rows will there be in my table? [Answer: $3024/24 = 126$.]

Note: We can issue 126 codewords at most, by taking exactly one codeword from each row. It is much easier to imagine an intelligence agency with 126 secret agents than one with 10 secret agents.

You might be curious if we can generalize the previous calculations to cases where we use any number of letters, or for codewords of any particular length.

Suppose the alphabet has n letters in it, and each codeword is x letters long. In general, if each row has b codewords and there are a codewords in the table, then there must be a/b rows. As we noted earlier, $a = P_{n,x}$ and $b = x!$, which means that the number of rows will be

$$\frac{a}{b} = \frac{P_{n,x}}{x!}$$

We will call this quantity $C_{n,x}$. One way to compute $C_{n,x}$ is

$$C_{n,x} = \frac{P_{n,x}}{x!}$$

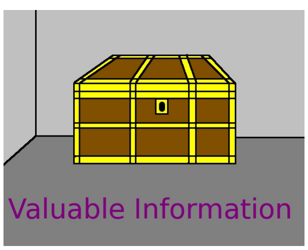
but we can also plug in the equation from the permutation principle

$$P_{n,x} = \frac{n!}{(n-x)!}$$

and obtain a different formula for the combination principle

$$C_{n,x} = \frac{P_{n,x}}{x!} = \frac{P_{n,x}}{1} \cdot \frac{1}{x!} = \frac{n!}{(n-x)!} \cdot \frac{1}{x!} = \frac{n!}{(n-x)!x!}$$

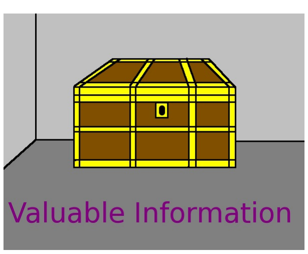
Moreover, most modern scientific calculators have a $C_{n,x}$ button that will permit you to calculate without the use of any formula at all.



When one must make x selections from a set of n options, order does not matter, and repeats are not allowed, then the number of possibilities is $C_{n,x}$. This is the combination principle.

There are two ways to compute $C_{n,x}$. They are

$$C_{n,x} = \frac{P_{n,x}}{x!} = \frac{n!}{(n-x)!x!}$$



The combination principle, as defined in the previous box, is the third member of a four-entry chart. This chart is phenomenally useful in solving combinatorial problems. We saw this chart for the first time on Page 998, and again on Page 1023.

	Repeats OK	No Repeats
Order Matters	Exponent Princ.	Permutation Princ.
Order Doesn't Matter	???	Combination Princ.

The final entry in that chart (order doesn't matter but repeats are allowed) happens to essentially never come up in application problems. For the sake of mathematical completeness, we will briefly discuss it on Page 1113.

Our next example is a bit simple as well, but we'll move on to harder examples soon afterward. This next example is meant to highlight the distinction between permutations and combinations. That's very important, because it will help you decide whether to use permutations or combinations when solving a problem.

For Example :

Let's suppose that five students (Alice, Bob, Charlie, Dave, and Ellen) want to form a new campus club—Future Actuaries of America. The student-government association requires them to choose a president, a vice-president, and a treasurer. Of course, no student can hold more than one office. As you might guess, being eager and motivated college students, all five students would like to be officers—plus it looks good on the resume. With that in mind, they decide to choose officers by drawing names out of a hat. The first name will be the president, the second name will be the vice-president, and the third name will be the treasurer. Suppose we wish to know, "In how many ways can officers be chosen?"

We will solve this problem in the next box.

7-9-4

To solve the problem of the previous box, we must address the two fundamental questions in a combinatorics problem: Does order matter? Are repeats allowed?

It is very clear that order matters in this example, because if we draw Charlie-Ellen-Alice then Charlie is the president, but if we draw Ellen-Charlie-Alice, then Ellen is the president. Those are two different situations! Therefore, order matters. Repeats cannot be permitted, because if we drew Bob-Bob-Dave, then Bob would be simultaneously president and vice-president, which was expressly forbidden in the previous paragraph.

Since order does matter and repeats are forbidden, we know that we should use the permutation principle. Using it, we compute that there are

$$P_{5,3} = \frac{5!}{2!} = \frac{120}{2} = 60$$

ways to choose officers in this problem. We will now continue in the next box.

For Example :

Let's revisit the five students of the previous box, a few months later. The five students receive an invitation from the national office of the Future Actuaries of America. Their club has been invited to send three students to a weekend conference in Miami, at the national office's expense. Unsurprisingly, all five want to go to the conference. To avoid an argument, they will decide the matter by drawing names out of a hat. We wish to know "In how many ways can a conference delegation of three officers be chosen?"

We will solve this problem in the next box.

7-9-5

Again, to solve the problem of the previous box, we must address the two fundamental questions: Does order matter? Are repeats allowed? Surely it does not make any sense at all to say that the three students selected for the conference are Bob, Alice, and Bob. Therefore, repeats cannot be allowed. Now let's address the question of whether or not order matters. If we draw the names Ellen-Charlie-Dave, or if we draw the names Charlie-Dave-Ellen, can these count as two different situations? In those two cases, the same set of students are going on the trip. It is the same situation. Therefore, order clearly does not matter in this example.

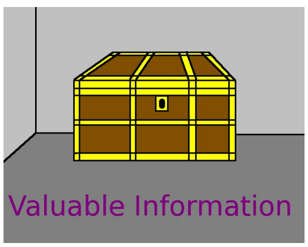
Since order does not matter and repeats are forbidden, we know that we should use the combinations principle. There are

$$C_{5,3} = \frac{5!}{2!3!} = \frac{120}{(2)(6)} = 10$$

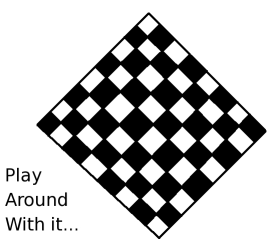
ways to choose a conference delegation in this problem.

Reflecting on the last two examples, let's re-examine the table from Page 1046 that had 60 entries (ABC, ACB, and so forth). Before, the entries in that table represented codewords for identifying secret agents. Now, the letters represent the five students from the previous two examples.

- If we look at the entries themselves, each group of three letters reflects one selection of officers. For example, BEA signifies that Bob is president, Ellen is vice-president, and Alice is the treasurer. Therefore, we're not surprised that we computed 60 different ways to select officers, because there are 60 entries in that table.
- If we look at the rows, each of the ten rows represents a delegation that could be sent to that conference. Each of the six entries in a particular row represent the same delegation, since order does not matter. If you look at two entries in different rows, you will see that a different subset of the students is being selected for the conference.



In general, whenever you are choosing a subset from a set, it is extremely likely that you are using the combinations principle. That's because the order of the entries in a set does not matter, and entries in a set cannot be repeated. Similarly, the combinations principle requires that order does not matter, and that repeats are not allowed.



7-9-6

- Suppose that a club has twelve members, and that they need to select four officers. Rather than have an election (since everyone is eager to be an officer), they decide to choose their officers by drawing names from a hat. In how many ways can officers be selected? [Answer: $P_{12,4} = 11,880$ ways.]
- Next, suppose that the same club is invited to send four members to a conference at the national office's expense. Since everyone wants to go, they decide to choose by drawing names from a hat. In how many ways can delegates be selected? [Answer: $C_{12,4} = 495$ ways.]

It is important to mention that some textbooks use different notation for the combinations operator. The following five are all equivalent.

$$C_{6,3} = C^6_3 = 6C3 =_6 C_3 = \binom{6}{3}$$

It is a bit unfortunate that there are so many different ways to write the same operation. In this textbook, we will always write $C_{6,3}$. On the bright side, there is only one way to write a factorial (e.g. 3!) and only four ways to write a permutation:

$$P_{6,3} = P^6_3 = 6P3 =_6 P_3$$

As you have probably noticed, we will always write $P_{6,3}$ in this textbook.

Play Around With it...

7-9-7

Let's suppose that a new handheld gaming gadget has nine possible attachments. During late summer and early fall, the warehouse manager wants to preassemble the gadgets, with their attachments, and store them in large bins. This way, they'll be able to ship them out rapidly when the December holidays approach. Every allowable arrangement of attachments will have a bin. Note, for this particular gadget, it doesn't make sense for the same attachment to be chosen more than once.

- The standard version comes with two attachments. How many possible choices for two attachments are there? [Answer: 36.]
- The deluxe version comes with three attachments. How many possible choices for three attachments are there? [Answer: 84.]
- How many bins are required, for storing all possible arrangements of attachments, for these standard and deluxe versions? [Answer: 120.]
- If each bin is 18 inches wide, and if they are stacked 4 high, how many linear feet of warehouse space do we need for the bins for these standard and deluxe versions? [Answer: 45 feet.]
- Note, for the benefit of our foreign students, there are 12 inches in a foot.

Play Around With it...

7-9-8

Let's suppose you work for a laptop manufacturer. Their new model of laptop can come with 1, 2, or 3 peripheral devices, chosen from a list of twelve choices. As in the previous box, it doesn't make sense to choose the same device twice. How many possible choices of peripherals are possible? [Answer: 298.]

Hint: I added together three different numbers to get 298.

Play Around With it...

7-9-9

Let's suppose among all three shifts at some factory, there are 37 lathe operators and 5 welders. There are some serious safety concerns and the workers want to approach management. It is normal (in such situations) to pick a small group because management might be intimidated if 42 workers enter the boss's office, all at once, with complaints. They decide to randomly select a delegation of five by drawing names out of a hat.

- In how many ways can a delegation of five be chosen from among all 42 workers? [Answer: $C_{42,5} = 850,668$ ways.]
- In how many ways can a delegation of five be chosen only from among the 37 lathe operators? [Answer: $C_{37,5} = 435,897$ ways.]
- In how many ways can a delegation of five be chosen only from among the 5 welders? [Answer: $C_{5,5} = 1$ way.]

For Example :

\# 7-9-10

We're going to further explore the situation in the previous box now. Imagine that the 42 workers have just put their names into the hat, and five names have been drawn. It is rather possible that the delegation of five might include zero welders. This might alarm the welders, because there could be some safety issues that are of concern to welders which lathe operators wouldn't know about, because the lathe operators are not welding. What if the drawing was rigged? What if the lathe operators had colluded, in some way, to ensure that no welders were included? In situations like this, it is useful to compute the probability that the delegation consists entirely of lathe operators, with zero welders, by coincidence.

That probability is given by the number of ways to make the delegation, using only lathe operators, divided by the number of ways to make the delegation from among all 42 workers. In other words, the universal set of this problem is the set of all possible delegations, drawn from all possible 42 workers. (That's the denominator.) We are interested in those delegations that are drawn only from lathe operators. (That's the numerator.)

$$\frac{C_{37,5}}{C_{42,5}} = \frac{435,897}{850,668} = 0.512417\cdots$$

As you can see, there is nothing surprising about the fact that 0 welders were selected.

For Example :

\# 7-9-11

Continuing with the situation of the last two boxes, let's now imagine that all 42 workers have put their names into the hat, but the drawing revealed only welders, and no lathe operators. Doesn't this seem strange, since 37 out of 42 of the workers are lathe operators? Let's compute the probability that the delegation is entirely composed of welders, by coincidence.

Again, the universal set of this problem is the set of all possible delegations, drawn from all possible 42 workers. (That's the denominator.) We are interested in those delegations that are drawn only from welders. (That's the numerator.)

$$\frac{C_{5,5}}{C_{42,5}} = \frac{1}{850,668} = 0.00000117554\cdots$$

It is very clear that in this situation, the lathe operators should at least suspect the possibility of foul play.

For Example :

\# 7-9-12

Continuing with the last few boxes, a slightly harder question is to compute the probability that the delegation is mixed. On the way to computing that probability, let's consider three events:

- The delegation consists entirely of lathe operators. Call this event $\mathcal{L}$.
- The delegation consists entirely of welders. Call this event $\mathcal{W}$.
- The delegation is mixed. Call this event $\mathcal{M}$.

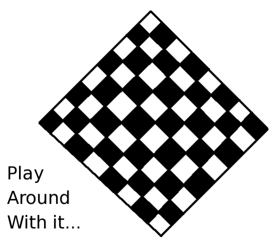
I'm going to claim that the events $\mathcal{L}$, $\mathcal{W}$, and $\mathcal{M}$ form a sample space. First, I should argue that they are mutually exclusive. That means there must be no circumstance where more than one of these three are true. Second, I should argue that they are collectively exhaustive. That means there must be no circumstance which is not covered among the three. Both of those requirements are obviously satisfied here. Yet, you might ask, "What is the consequence of applying the rules of a sample space to this problem?"

The key is to realize that the probabilities in a sample space add to one. We can use this to find the probability of $\mathcal{M}$ given the probability of $\mathcal{L}$ and $\mathcal{W}$. We will do that in the next box.

Continuing with the previous box, we know the probabilities of $\mathcal{L}$ and $\mathcal{W}$ from the previous few boxes. Let's denote the probability of $\mathcal{M}$ as x. We can solve for x by setting the sum of these probabilities to one.

$$\begin{aligned} Pr\{\mathcal{L}\} + Pr\{\mathcal{W}\} + Pr\{\mathcal{M}\} &= 1 \\ 0.512417 + 0.00000117554 + x &= 1 \\ x &= 1 - 0.512417 - 0.00000117554 \\ x &= 0.487581\cdots \end{aligned}$$

Now we know the probability of a mixed delegation is $0.487581\cdots$.

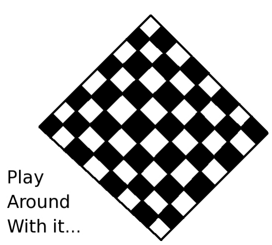


7-9-13

Let's imagine that a group of 16 engineers are working under a contract with a firm in China. The Chinese firm would like to offer the engineers a tour of their facilities, but it is very expensive to fly across the Pacific Ocean, and they therefore offer to pay for only 5 engineers to visit. One of the computer engineers, Lars, who doesn't want to go, has written a computer program that he says will select 5 people at random from a set of 15 people. Of the 15 remaining engineers, there are 10 from Minnesota and 5 from Wisconsin. Yet, all 5 of the winners were from Minnesota. Lars is accused of rigging the computer program to only select Minnesotans, as he is a Minnesotan himself.

- In how many ways can 5 travelers be selected from the set of 15 engineers? [Answer: $C_{15,5} = 3003$.]
- In how many ways can 5 travelers be selected from the set of 10 Minnesotan engineers? [Answer: $C_{10,5} = 252$.]
- In how many ways can 5 travelers be selected from the set of 5 Wisconsinite engineers? [Answer: $C_{5,5} = 1$.]

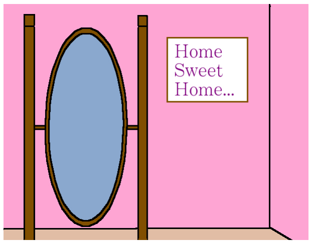
We will continue in the next box.



7-9-14

Continuing with the previous box,

- If the names were truly selected at random among all 15 engineers regardless of origin, then what would the probability be that all the selectees were all from Minnesota, by coincidence? [Answer: $252/3003 = 0.0839160\cdots$.]
- If the names were truly selected at random among all 15 engineers regardless of origin, then what would the probability be that all the selectees were all from Wisconsin, by coincidence? [Answer: $1/3003 = 0.000333000\cdots$.]
- If the names were truly selected at random among all 15 engineers regardless of origin, then what would the probability be that both states are represented? [Answer: $0.915750\cdots$.]

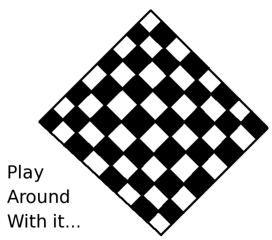


A Pause for Reflection. . .
Observe that I did not ask you to pass judgement on Lars in the previous box. Questions like, "What is the probability that all the selectees are Minnesotan by coincidence?" have a unique and clearly identifiable answer. In this case the answer was 0.0839160, though the answers 0.0839159 or 0.0839161 could occur due to rounding error. Any other answer whatsoever would be wrong.

Questions involving judgement and appraisal are not mathematical. However, there is a standard that many Federal investigators use in the USA. That standard is 1%, but one must be very careful with how that standard is used.

In the previous box, the probability that all the selectees were Minnesotan was 8.39%, which is much larger than 1%. That means we have no right to be surprised that all the selectees were Minnesotan. However, the probability that all the selectees were Wisconsinites was 0.03%, which is much smaller than 1%. In that situation, further investigation is merited. For example, we might look at the source code of Lars's program, especially if it was a short program and if it was written in a commonly understood computer language like `C`, `Java`, or `Python`. Then, any computer programmer would be able to read all the lines of the program, and pass judgement.

What cannot be done, under any circumstances, is to use the probability calculations alone to judge. After all, rare events do sometimes occur.



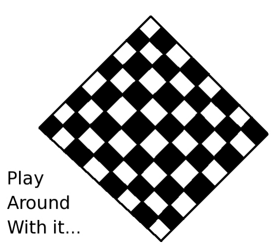
7-9-15

The senior and junior vice-presidents in charge of expansion at a NYC-area grocery store are tasked with determining which sites in New Jersey, New York, and Connecticut are suitable for new stores. They are considering 17 sites at this point, including 3 in NJ, 8 in NY, and 6 in CT. However, there is only enough funding for 8 sites, so they are tasked with narrowing down the list.

As it comes to pass, the senior vice-president is from NJ and the junior vice-president is from CT. Therefore, when it is revealed that none of the eight new stores will be in NY, there is some suspicion of anti-NY bias.

- In how many ways could 8 stores be chosen using only NJ and CT? [Answer: 9.]
- In how many ways could 8 stores be chosen using all three states? [Answer: 24,310.]
- What is the probability, if the sites were chosen entirely at random, that zero NY sites would be selected? [Answer: $0.000370218\cdots$.]

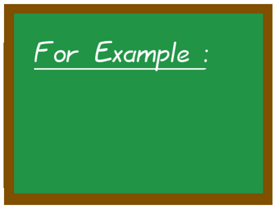
Note, this isn't evidence of foul play by itself. There could have been several reasons to rule out all the NY sites. Perhaps it could be something about the taxes, which vary from state to state. Alternatively, perhaps it could be that they already have lots of stores in NY and the vice-presidents are worried about market saturation.



7-9-16

Let's revisit the problem of the previous box. Suppose it had been the case that the list of new store locations included sites in NJ and sites in NY, but zero sites in CT.

- In how many ways could 8 stores be chosen using only NJ and NY? [Answer: 165.]
- In how many ways could 8 stores be chosen using all three states? [Answer: 24,310.]
- What is the probability, if the sites were chosen entirely at random, that zero CT sites would be selected? [Answer: $0.00678733\cdots$.]



7-9-17

You might have noticed that at the end (or start) of a long meeting, that the amount of time required for everyone to shake everyone else's hand can be significant, especially if there are a lot of people in the room. Suppose there are six people on the hiring committee, but twelve people on the website-design committee. (These committees are entirely separate.) For each committee meeting, how many handshakes are there in total, when everyone shakes everyone else's hand? Furthermore, how many hands does each person shake?

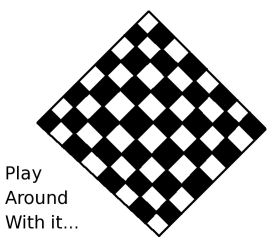
Each hand shake consists of two people. Since there are six people on the hiring committee, then $C_{6,2} = 15$ possible pairs of people can be listed, and therefore 15 handshakes will occur. Similarly, since there are twelve people at the website oversight committee, then $C_{12,2} = 66$ possible pairs of people can be listed, and therefore 66 handshakes will occur.

We will continue this problem in the next box.

Let's continue with the previous box. Of course, if there are six people in a room, then each person must shake five hands. However, $(6)(5) = 30 \neq 15$. How can this be? The answer is that each handshake involves two people. We can see that $(15)(2) = 30$ and therefore everything makes sense.

Likewise, if there are twelve people in a room, then each person must shake eleven hands. However, $(12)(11) = 132 \neq 66$. We must remember, yet again, that each handshake involves two people. We can see that $(66)(2) = 132$ and therefore everything makes sense.

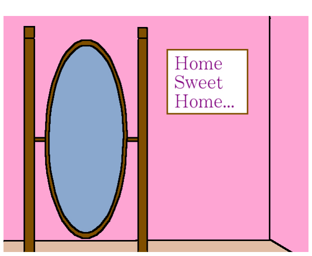
This is a great way to check your work.



7-9-18

Let's suppose that there is a meeting with the following number of people in it. How many handshakes will there be?

- If there are four people? [Answer: 6 handshakes.]
- If there are eight people? [Answer: 28 handshakes.]
- If there are ten people? [Answer: 45 handshakes.]



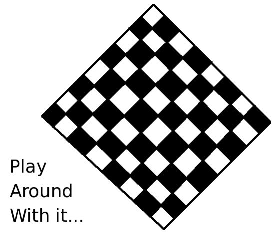
A Pause for Reflection. . .

Have you ever noticed that in social situations of a long duration, such as a multi-day road trip or perhaps sharing an apartment for a semester, that very small groups rarely have any conflict, but larger groups have a high risk of two people entering into conflict?

I think the previous three boxes explain one reason why this appears to be true. Imagine that each time a pair of people shake hands, that somewhere in another universe, a computer is giving them each a personality test and seeing if they are socially compatible or socially incompatible. If there are only four people, a mere 6 "compatible" results must come out for a quiet social environment. At the other extreme, if there are only twelve people, a whopping 66 "compatible" results must come out, and that's not the same as 6.

The following table summarizes this notion:

Number of people	4	6	8	10	12
Number of pairs	6	15	28	45	66



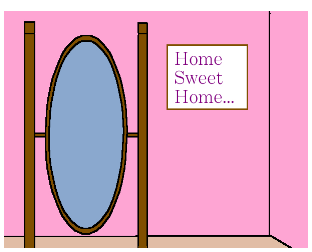
7-9-19

Imagine that the chess club and the boxing club each want to have a tournament, where everyone in the club faces off against everyone else. The chess club has 15 members and the boxing club has 40 members.

- How many chess games must be played? [Answer: 105 games.]
- How many boxing matches must be boxed? [Answer: 780 boxing matches.]
- Now further suppose that the boxing club invites the members of the chess club to join in the boxing competition. How many boxing matches must be played, in total, if everyone from each club boxes with every one from both clubs?
[Answer: 1485 boxing matches.]

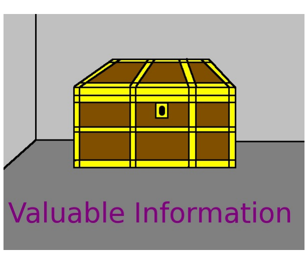
We can check our work from the previous box, using the strategy from the handshakes.

- Each of 15 chess club members must play chess against 14 other people. First, we compute $(15)(14) = 210$ and then, we compute $(105)(2) = 210$. That's a match!
- Each of 40 boxing club members must box 39 other people. First, we compute $(40)(39) = 1560$ and then, we compute $(780)(2) = 1560$. That's a match!
- Now imagine a combined club, with $15 + 40 = 55$ members. Each member of the combined club must box with 54 people, so we compute $(55)(54) = 2970$. Next, we compute $(1485)(2) = 2970$. That's a match!



A Pause for Reflection. . .
In the previous checkerboard box, I made an unfair assumption. Can you spot it?

Technically speaking, I have assumed that the boxing club and the chess club had no overlapping members. In my experience, this is often the case. However, my good friend Prof. K. Wojciechowski is both a chess player and a boxer (as well as a mathematics professor and a musician), so we must be careful not to make assumptions based on stereotypes.



We've just now seen several examples where the solution to a problem is given by $C_{n,2}$, for some value of n. That's when you want to see how many possible pairs can be constructed from n persons or objects. In reality, this is just a special case of the combinations principle; yet, many books, including this one, call that *the handshake principle*.

Among n people or objects, there are

$$C_{n,2} = \frac{n(n-1)}{2}$$

possible pairs of people or objects.

Most students find it easier to just remember the combinations formula. However, the shortcut formula in this box is occasionally useful. In the next box, we'll see an application of this idea to data mining.

Data Mining involves looking at large data sets, with many variables, to discover relationships—including relationships that you did not even suspect existed. For example, each cellular phone service provider has hundreds of millions of records, one for every single text message and every phone call made by anyone in their network each year.

When engaging in data mining, one of the simplest tools is the "correlation coefficient," usually called r. This tool can be used to measure how correlated two variables are, e.g. cholesterol and the frequency of heart attacks. The formula for r is a bit complicated, but it can be computed for any two variables in a data set.

Therefore, it is useful to compute how many "pairs of variables" a data set will have.

Play Around With it...

7-9-20

Let's consider some data mining situations, listed below, and compute how many pairs of variables there are.

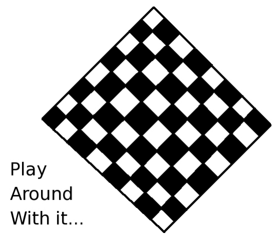
- Suppose it is health data, with 20 variables per patient. How many correlation coefficients can be computed?
  [Answer: 190.]
- Suppose it is data about aircraft for an airline, to help them identify the root causes of some maintenance issues. Suppose there are 100 variables per aircraft. How many correlation coefficients can be computed?
  [Answer: 4950.]
- Suppose it is data about genomics, looking at human DNA to make a test to predict colon cancer. Suppose there are 10,000 variables per sample. How many correlation coefficients can be computed?
  [Answer: 49,995,000.]

Now you can understand why some of the most expensive computing resources in the world—cloud computing systems comprised of hundreds of servers scattered around the world—are frequently put to use to tackle large data mining problems. There are many other applications besides those listed in this box.

Cryptographic systems for encryptions come in two broad classes: public-private key systems which are sometimes called public key systems, and secret key systems, which are sometimes called symmetric key systems. The public-private key system is what is used today, and it was absolutely crucial for the development of e-commerce on the internet. We do not have time to go into that now, unfortunately, but every business person should try to be familiar with it.

Prior to the invention of public-private key systems, the collection of systems that were used (called "secret key systems") required every pair of people who wanted to communicate to exchange a "secret key." The technical term "secret key" here is a string of 0s and 1s, and it is very important to keep it secret. (We learned about this on Page 999.) In fact, it is even more important to keep secret keys protected than passwords. That's because if you accidentally share your password with others (for example, because you are drunk at a party) then you can always change it. However, if a secret key gets released, especially on the web, then all the messages that were encrypted with that key will now become readable by anyone whatsoever. There is no remedy or mechanism to undo the release of a secret key.

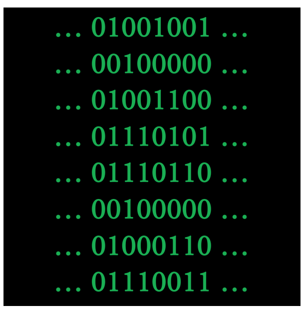
As you can see, secret keys have to be protected, to safe guard private information. We're going to explore how impractical this can get as a company grows, in the next box.



7-9-21

We will consider a new "start up" company, and analyze what happens if they use symmetric key cryptography. In particular, we'll compute the number of secret keys that they would need as the company grows in size. In order for it to be the case that any two employees can exchange encrypted data or have an encrypted conversation, as it turns out, there must be one secret key for every pair of employees.

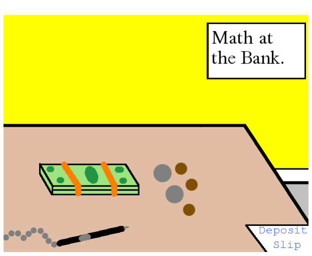
- When the company has 8 employees, how many secret keys are needed? [Answer: $C_{8,2} = 28$ secret keys.]
- When the company has 80 employees, how many secret keys are needed? [Answer: $C_{80,2} = 3160$ secret keys.]
- When the company has 800 employees, how many secret keys are needed? [Answer: $C_{800,2} = 319,600$ secret keys.]
- When the company has 8000 employees, how many secret keys are needed? [Answer: $C_{8000,2} = 31,996,000$ secret keys.]



Look at the numbers in the previous box carefully. As you can see, as the company grows, the number of secret keys required is growing out of control. That's why companies are careful to use public-private key systems.

In public-private key systems, each employee has exactly two keys, a public key and a private key. In that case, 80 employees would require 160 keys, 800 employees would require 1600 keys, and 8000 employees would require 16,000 keys.

It is much more practical!

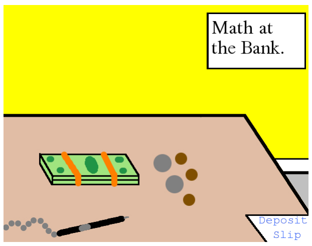


Let's talk about a managerial topic for a moment. There was an extremely influential book in the technology sector, called *The Mythical Man Month: Essays on Software Engineering*, by Fred Brooks, published in 1975 and again in 1995.

As it comes to pass, Brooks was the manager assigned to a large project at IBM, designing an entire operating system! The project was behind schedule. He decided to add programmers to the project, which (counter-intuitively) made the project even later.

First, let's look at why he thought adding additional programmers would help. If you're managing people who are laying bricks, stuffing envelopes, or chopping down trees, you could calculate how many "man-months" (we would say "worker-months") of work there happens to be. Then you can add workers to do the work faster. For example, if there are 300 worker-months of work, and you go from 30 to 60 workers, then you would go from 10 months down to 5 months.

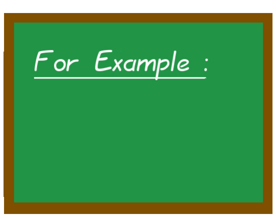
However, this is not true in software development. If you have 30 programmers, then there are $(30)(29)/2 = 435$ channels of communication. When you increase to 60 programmers, then there are $(60)(59)/2 = 1770$ channels of communication, which is far worse! Even more horrible would be 90 programmers, where there would be $(90)(89)/2 = 4005$ channels of communication. Imagine when a programmer has to go to all of his teammates, and ask "Would it break your parts of the code if I changed this thing here to that?" I'm sure that we can agree that 4005 channels (or even 1770 channels) of communication is much worse than 435 channels of communication. We will continue in the next box.



Continuing with the previous box, for the above reasons, a maxim was born: "Brooks' Law: Adding manpower to a late software project makes it later." A fascinating analogy—and one that is quoted much more often—is that "1 woman can make 1 baby in 9 months, and 9 women can make 9 babies in 9 months, but 9 women cannot make 1 baby in 1 month."

So, what should a manager do to avoid this paradox?

- First and foremost, the manager should choose *the correct number of programmers* from the beginning. This should not be changed, or it should be changed only in the first third of the project. In fact, just *on-boarding* new programmers takes some time. The process of on-boarding is explaining all of the components of the software that already exist, how they work, and how they interact with each other, and then carving up the remaining tasks between all the programmers. During that on-boarding period, the new programmers cannot yet work, and are therefore idle, and the old programmers cannot work, because they are busy on-boarding the new people. Sometimes, on-boarding is called *ramp up*.
- Second, the promises of delivery dates (the day when a piece of running code will be ready and tested) should build in large margins of error, to protect against delays that might occur. For example, one of my favorite bosses when I worked at the NSA would ask technical people how long something took. If someone said 4 days, she'd report to the next higher level 2–16 days. Similarly, if someone said 90 days, she'd report to the next higher level 45–360 days.
- Third, if a chunk of software is running late, the best strategy is to just write a new timeline for the project, or to move forward with the entire project not using that particular chunk of software causing the delay. Interfering with computer programmers is only going to cause damage.
- Programmers are not alone in this respect. Once an open-heart surgery has begun, the management team of a hospital should not be adding or removing surgeons from that surgery. The team should be left to complete their project/surgery, unmolested.



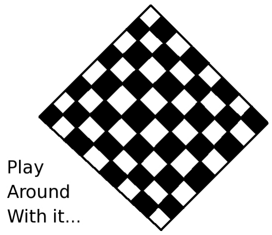
7-9-22

Let's look at an easy but cool pattern now.

$$\begin{aligned}
1 &= 1 = 2/2 = (2)(1)/2 = C_{2,2} \\
1+2 &= 3 = 6/2 = (3)(2)/2 = C_{3,2} \\
1+2+3 &= 6 = 12/2 = (4)(3)/2 = C_{4,2} \\
1+2+3+4 &= 10 = 20/2 = (5)(4)/2 = C_{5,2} \\
1+2+3+4+5 &= 15 = 30/2 = (6)(5)/2 = C_{6,2} \\
1+2+3+4+5+6 &= 21 = 42/2 = (7)(6)/2 = C_{7,2} \\
1+2+3+4+5+6+7 &= 28 = 56/2 = (8)(7)/2 = C_{8,2} \\
1+2+3+4+5+6+7+8 &= 36 = 72/2 = (9)(8)/2 = C_{9,2} \\
1+2+3+4+5+6+7+8+9 &= 45 = 90/2 = (10)(9)/2 = C_{10,2} \\
1+2+3+4+5+6+7+8+9+10 &= 55 = 110/2 = (11)(10)/2 = C_{11,2} \\
1+2+3+4+5+6+7+8+9+10+11 &= 66 = 132/2 = (12)(11)/2 = C_{12,2}
\end{aligned}$$

As you can see, the sum of the first n integers is just a handshake principle.

$$1+2+3+4+\cdots+(n-2)+(n-1)+n = C_{n+1,2} = \frac{(n+1)(n)}{2}$$

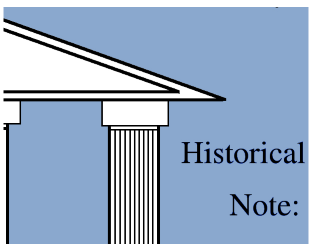


7-9-23

There is a legend that when Johann Carl Friedrich Gauss (1777–1855) was in elementary school, the teacher was frustrated with how much faster he was learning math, compared to his classmates. The teacher then told Gauss to sum the first 1000 positive integers. He was done in a shockingly short amount of time. It is a nice story, because Gauss easily ranks in the "top ten mathematicians" of all time. In fact, he has so many accomplishments, I won't list them here.

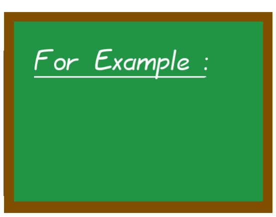
What is the sum of the first 1000 integers? [Answer: $(1001)(1000)/2 = 500,500$.]

As you can see, it doesn't take much time at all to compute that sum.



The box above contains one of the most often repeated stories in all of mathematics. Unfortunately, it is really obvious that the story about Gauss is false. Instruction was not done in classrooms when Gauss was growing up. Instruction was always 1-on-1 at that time. The idea of a classroom, and the instruction of numerous young children in bulk, came much later (in the mid-to-late 19th century). That's an error of about ninety years, or four generations.

Sometimes, the story is even said about Isaac Newton (1642–1726), who invented large pieces of calculus and physics. There, the situation is even more absurd, because of the dates involved. It is really rather sad that this story is often told.



7-9-24

Let's imagine that in some "Little League" for baseball in a major city, we discover that there were 1081 games. Furthermore, we're told that every team has played every other team exactly once. Can we determine how many teams there are?

Of course, we could just guess various values of n, and check $C_{n,2}$ until we find the n that results in $C_{n,2} = 1081$. However, it turns out that we can solve for this more explicitly. We should use the shortcut formula for the handshake principle. We know that $C_{n,2} = n(n-1)/2$, so we just set that equal to 1081, and solve for n.

$$\begin{aligned}
C_{n,2} &= 1081 \\
\frac{n(n-1)}{2} &= 1081 \\
n(n-1) &= 2162 \\
n^2 - n &= 2162 \\
n^2 - n - 2162 &= 0 \\
n &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2162)}}{2(1)} \\
n &= \frac{1 \pm \sqrt{1 + 8648}}{2} \\
n &= \frac{1 \pm 93}{2} \\
n &= 94/2 \text{ or } -92/2 \\
n &= 47 \text{ or } -46
\end{aligned}$$

Since a negative number of teams would not make sense, clearly there are $n = 47$ teams.

Play Around With it...

\# 7-9-25

Suppose that your company hires a very expensive consulting firm to analyze some data from all the sales throughout your company over the last several years. They brag, in their presentation, at having analyzed 15,051 pairs of variables during the data mining. How many variables did they consider?
[Answer: 174.]

For Example :

\# 7-9-26

Let's suppose that I'm going on a business trip to Duluth, perhaps to do a book-signing for one of my textbooks. Unless it is June, July, or August, I'll need to bring sweaters. Perhaps I have narrowed it down to seven nice sweaters, but I only need to take three. How many ways are there for me to choose three sweaters from among seven to put into my bag?

Surely, order doesn't matter, because I'm just putting them into my bag. I can't choose the same sweater more than once, so repeats are not allowed. Then the answer must be

$$C_{7,3} = 35$$

Yet, surely choosing which 3 to take with me is the same act as choosing which 4 to leave behind. If you know which 4 I have left behind, then surely you know which 3 I have brought with me. What happens if I write $C_{7,4}$ on an exam instead of $C_{7,3}$? Well, we can ask our calculator and we find out

$$C_{7,4} = 35$$

as well. Therefore, there is no problem.

For Example :

\# 7-9-27

Imagine that an elementary school decides to take all its third-graders on a trip to the zoo. There are 30 kids who can go, and that's a bit of a larger group than the zoo is used to. The office at the zoo suggests that they can split the group into two subgroups. One subgroup, of 20 kids, can go to see the monkeys. The other subgroup, of 10 kids, can go to see the penguins. The teachers find this out the day before the trip so they decide to randomly select which kids get to go to the monkeys and which get to go to the penguins. Otherwise, they might have to settle a myriad of disputes if one or the other type of animal happens to be very popular. How many possible splittings are there of the 30 kid group into a 20 kid subgroup and a 10 kid subgroup?

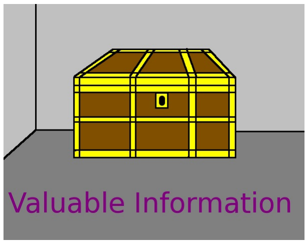
One way to count that is to say that out of 30 kids, you are selecting 20 to go see monkeys. Order doesn't matter, and you can't select the same kid twice, so there are no repeats. With that in mind, we are definitely using combinations. We obtain

$$C_{30,20} = 30,045,015$$

Alternatively, another way to count that is to say that out of 30 kids, you are selecting 10 to go see penguins. We obtain

$$C_{30,10} = 30,045,015$$

I suppose it is surprising that the numbers come out the same, but it also makes sense. The act of selecting 20 to see the monkeys is the same act as selecting 10 to see the penguins.



The property that I have demonstrated in the previous two examples generalizes. One way to write it is that

$$C_{a+b,a} = C_{a+b,b}$$

Another way to write it is

$$C_{n,x} = C_{n,(n-x)}$$

Let's check if the formulas in the previous box match the previous two examples.

- Using $C_{a+b,a} = C_{a+b,b}$, for the sweaters needed in Duluth, we can set $a = 3$ and $b = 4$. Then we have $C_{7,3} = C_{7,4}$.
- Using $C_{n,x} = C_{n,(n-x)}$, for the sweaters needed in Duluth, we can set $n = 7$ and $x = 3$. Then we have $C_{7,3} = C_{7,4}$.
- Using $C_{a+b,a} = C_{a+b,b}$, for the kids going to the zoo, we can set $a = 10$ and $b = 20$. Then we have $C_{30,10} = C_{30,20}$.
- Using $C_{n,x} = C_{n,(n-x)}$, for the kids going to the zoo, we can set $n = 30$ and $x = 10$. Then we have $C_{30,10} = C_{30,20}$.

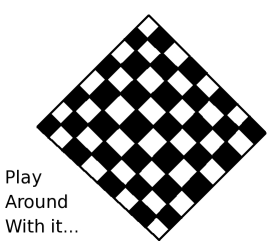
We can actually prove those two formulas, with simple algebra.

For the first one,

$$C_{a+b,a} = \frac{(a+b)!}{a!(a+b-a)!} = \frac{(a+b)!}{a!b!} = \frac{(a+b)!}{(a+b-b)!b!} = C_{a+b,b}$$

For the second one,

$$C_{n,(n-x)} = \frac{n!}{(n-x)!(n-(n-x))!} = \frac{n!}{(n-x)!(n-n+x)!} = \frac{n!}{(n-x)!(x)!} = C_{n,x}$$



7-9-28

Problems about pizza toppings are a classic in courses such as this one. As you can see, in this module, we've had problems about cryptographic keys and data mining, so you already know that combinations are relevant to practical matters. In fact, we'll shortly see some problems about investing, another important topic. Therefore, it would be silly for us to waste space discussing pizzas in detail. Nonetheless, for the sake of tradition, we will have a question about pizza toppings.

Suppose a particular pizzeria offers 12 possible toppings. For the moment, assume that the order of the toppings does not matter.

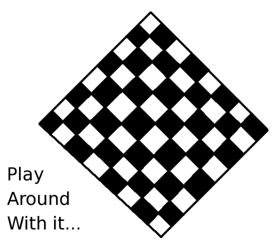
- How many possible pizzas are there with three toppings? [Answer: 220.]
- How many possible pizzas are there with two toppings? [Answer: 66.]
- How many possible pizzas are there with one topping? [Answer: 12.]
- How many possible pizzas are there with at most three toppings? [Answer: 299.]

Hint: Did you remember to include the option of a 0-topping pizza?

Of course, the only way that we have the right to use the combinations formula to solve the problem in the previous box is if we are confident that the order of pizza toppings doesn't matter. If the order of the pizza toppings matters, then we must use permutations. All the finite mathematics books that I have checked do these problems using combinations, thus tacitly claiming that order doesn't matter.

I agree in most cases that order doesn't matter. However, in some cases, it might matter. Consider "pepperoni above extra cheese" versus "pepperoni below extra cheese," or perhaps "pepperoni above sausage" versus "pepperoni below sausage." In these cases, one topping might become invisible. The extra cheese would hide the pepperoni, or the pepperoni would hide the sausage. Therefore, the pizzas would be visually distinguishable from each other when the order of the toppings is changed. For this reason, we are forced to concede that in some cases, order does matter. Yet, in many cases, the order would not matter. It is extremely inconvenient that we do not have a definite "yes" nor a definite "no" for the question "Does order matter?"

In the final analysis, we should conclude that permutations and combinations are not the right tool for analyzing pizza toppings, unless the problem has explicitly stated that the order of the toppings does not matter.



7-9-29

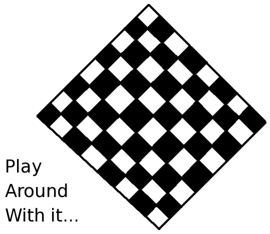
Returning to finance, a friend of mine once remarked at how much effort is put into choosing stocks and mutual funds for investment. It is true, that most investment banks assign the majority of their analysts to this one single task. Why is it so hard to select a good set of stocks?

To investigate this, it might be interesting to compute how many possible sets of six stocks can be picked, from each of the major indices. Just as a reminder, the Dow Jones Industrial Average (DJIA) has 30 stocks, while the S&P500 has 500 stocks, and the Russel2k has 2000 stocks.

- How many sets of six stocks are there on the DJIA? [Answer: 593,775.]
- How many sets of six stocks are there on the S&P500? [Answer: $2.10576\cdots \times 10^{13}$.]
- How many sets of six stocks are there on the Russel2k? [Answer: $8.82241\cdots \times 10^{16}$.]
- If an investor is open to purchasing stock from any of those three indices, how many possible sets of six stocks are there? [Answer: $3.62087\cdots \times 10^{17}$.]

It turns out that combinations really are all about choosing a set's subsets, but taking only those of a fixed size. The following questions recap the previous problems, but in the context of subsets.

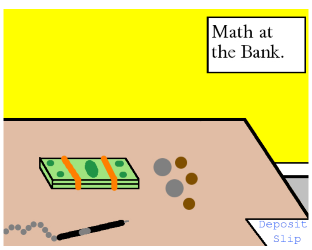
- How many subsets, from the set of 500 stocks on the S&P500, have six members?
- How many subsets, from the set of 12 pizza toppings, have three members?
- How many subsets, from the set of 30 students, have 20 members?
- How many subsets, from the set of 10 sweaters, have 3 members?
- How many subsets, from the set of 8000 employees, have 2 members?



7-9-30

In the largest gyms of major cities, it is common for there to be a few concession stands that sell healthy food. Suppose there is a successful juice bar that sells smoothies made from fruits and vegetables. However, the place is so popular, that there are often long lines, as patrons queue up for a smoothie. Now suppose that the manager decides that all combinations should be premixed, so that no one has to wait. The smoothies are always made from three ingredients.

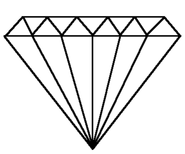
- If there are five ingredients to choose from, how many possible smoothies are there? [Answer: 10.]
- If there are six ingredients to choose from, how many possible smoothies are there? [Answer: 20.]
- If there are seven ingredients to choose from, how many possible smoothies are there? [Answer: 35.]
- The boss now sets up a suggestion box, and asks customers if they'd like to recommend any new ingredients for the smoothies. He gets so many suggestions that now there are 20 ingredients.
- If there are twenty ingredients to choose from, how many possible smoothies are there? [Answer: 1140.]



The previous box is a good example of scalability. Behind the counter, there can be room for 10 or even 20 pitchers of premixed fluid. That's fine, but it would be very difficult to find room for 35 pitchers. The plan was practical for small numbers of ingredients, but as the number of ingredients increases, the plan rapidly becomes unworkable.

This is made even more clear when we consider the possibility of 20 ingredients. Surely there cannot be room for 1140 pitchers of premixed fluid—that's absurd. Business plans or other technical designs are called scalable if they can handle growth easily. The situation in the previous box is an example of a situation that is not scalable.

Hard but Valuable!



Once a friend of mine, who has a PhD in physics, was studying for the actuary exams. (An actuary is a sort of "ultra-elite statistician" and you have to pass a sequence of very difficult exams to become one.) He came to me with a mathematics question. He observed that his calculator could not compute 100!, but it could compute $C_{100,50}$, so clearly there must be some way to find $C_{100,50}$ other than with

$$C_{100,50} = \frac{100!}{(50!)(50!)}$$

We will explore those "other methods" momentarily. This material is mathematically interesting and anyone who reads it will understand the concepts of this chapter better. That being said, very few instructors will want to cover this material.

Now, we ask ourselves, how did people compute with the combinations and permutations operators before the invention of cheap calculators? How about with the primitive 4-function calculators of the 1970s and 1980s? The techniques are actually rather interesting.

First, let's see why it is understandable that his calculator could not compute 100! at all. Using Sage, I can find that number. It is written below.

$$100! = 93,326,215,443,944,152,681,699,238,856,266,700,490,715,968,264,381,621,468,592,963,895,217,599,993,229,915,608,941,463,976,156,518,286,253,697,920,827,223,758,251,185,210,916,864,000,000,000,000,000,000,000,000$$

By the way, the command in Sage is `factorial(100)`.

Let's first examine what techniques would not work.

We could break $C_{15,5}$ into 15!, 10! and 5!, using the formula

$$C_{n,x} = \frac{n!}{(x!)(n-x)!}$$

which would give us

$$C_{15,10} = \frac{15!}{(10!)5!} = \frac{1,307,674,368,000}{(3,628,800)(120)}$$

but that's certainly not going to be fun in the era before calculators.

We could make use of the formula

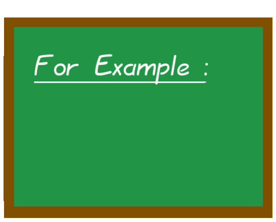
$$C_{n,x} = \frac{P_{n,x}}{x!}$$

which would give us the following

$$C_{15,10} = \frac{P_{15,10}}{10!} = \frac{(15)(14)(13)(12)(11)(10)(9)(8)(7)(6)}{(10)(9)(8)(7)(6)(5)(4)(3)(2)(1)}$$

which is not terrible (because there are cancellations) but perhaps we can do better.

We'll see a better way in the next box. We're going to combine these two ideas, which failed individually. Together, they will give us a successful way of solving this problem.

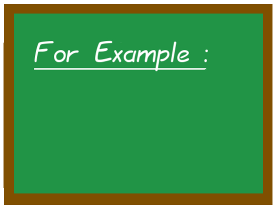


\# 7-9-31

Let's continue to explore how we might compute $C_{15,5}$ with a primitive hand calculator, as we might have done in the 1970s or 1980s.

First, we're going to break the combinations operator into three factorials. Second, we're going to pair up the two largest factorials, and make that into a permutation. The remaining factorial will become our denominator, whereas the permutation becomes the number. Observe,

$$C_{15,5} = \frac{15!}{(5!)(10!)} = \frac{15!}{10!} \cdot \frac{1}{5!} = \frac{P_{15,5}}{5!} = \frac{(15)(14)(13)(12)(11)}{(5)(4)(3)(2)(1)} = \frac{360,360}{120} = 3003$$



7-9-32

We can carry out the same strategy for $C_{20,7}$, as another example.

We will use the same strategy as the previous box,

$$C_{20,7} = \frac{20!}{(7!)(13!)} = \frac{20!}{13!} \cdot \frac{1}{7!} = \frac{P_{20,7}}{7!} = \frac{(20)(19)(18)(17)(16)(15)(14)}{(7)(6)(5)(4)(3)(2)(1)} = \frac{390,700,800}{5040} = 77,520$$

This is much easier than computing

$$20! = 2,432,902,008,176,640,000$$

and then performing two long divisions, one by $7! = 5040$ and another one by

$$13! = 6,227,020,800$$

In reality, the last two boxes were written in the style of the era of simple 4–6 function hand calculators. These could do addition, subtraction, multiplication, division, and then maybe percents and square roots, but that's it. In the completely pre-calculator era (i.e. the 1950s and earlier), one would take advantage of cancellations to avoid long divisions entirely, and reduce the problem to a handful of multiplications.

In the case of

$$\frac{(20)(19)(18)(17)(16)(15)(14)}{(7)(6)(5)(4)(3)(2)(1)} = (19)(17)(16)(15) = 77,520$$

we see that the (7) and (2) in the denominator take out the 14 in the numerator; similarly, the (6) with the (3), both from the denominator, take out the 18 in the numerator. Likewise, the (5) and the (4) in the denominator take out the 20 in the numerator. This leaves us with (19)(17)(16)(15). That's a lot easier than multiplying 19 integers together in order to obtain 20!, if we approached the problem naïvely.

In the case of

$$\frac{(15)(14)(13)(12)(11)}{(5)(4)(3)(2)(1)} = (7)(13)(3)(11) = 3003$$

the (5) and the (3) take out the 15 in the numerator; the (4) in the denominator turns the 12 into a 3. Finally, the 2 turns the 14 into a 7.

Believe it or not, it gets even neater. There is a computation trick, called Pascal's Triangle. Here are the first few rows of the triangle itself. We will explore it more, in the next box.

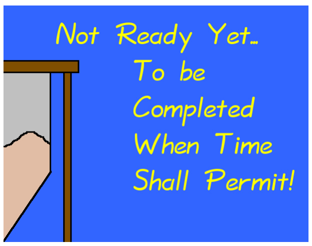
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Let's look at the triangle in the previous box. In particular, let's examine the line that begins with "1, 6, . . . ,"

This line actually gives you *all of the combinations* for "6" being the first number. In particular, we can confirm this particular row easily enough.

$$C_{6,0} = 1 \qquad C_{6,1} = 6 \qquad C_{6,2} = 15 \qquad C_{6,3} = 20$$
$$C_{6,4} = 15 \qquad C_{6,5} = 6 \qquad C_{6,6} = 1$$

You can confirm that this also works for any other row. Perhaps you might like to check the line that begins with "1, 5, . . . ," yourself.



I'm planning to write an entire module about Pascal's Triangle, but I haven't done that yet. It has a whole horde of properties, and applications, some of which have nothing to do with probability and combinatorics. It really is an amazing piece of mathematics.

However, since that module does not yet exist, we cannot proceed further at this time. (Actually, this module is currently a lecture and a worksheet in Math-270: *Discrete Mathematics*.)

This module is now complete. Thank you for reading.