

Module 8.1: Advanced Topics in Set Theory

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Overview

This assignment will expose you to some advanced topics of set theory, including some applications to number theory. That's kind of cool, as it ties together the first topic in this course (set theory) with our last major topic (number theory). As always in Math-270: *Discrete Mathematics*, this assignment will not be collected, checked, or graded. It is for your own development, and 1–2 questions from here will be on Biweekly Test #1. This took a really long time for me to write, so please read it and solve every problem.

Before we get into the questions, I have a warning about notation for you.

Warning about Dangerous Notation

Many computer engineers use the “overbar” to denote a complement. For example,

\mathcal{X}^c becomes $\overline{\mathcal{X}}$ $(\mathcal{A} \cap \mathcal{B})^c$ becomes $\overline{\mathcal{A} \cap \mathcal{B}}$ $\mathcal{X} \cap (\mathcal{Y} \cup \mathcal{Z})^c$ becomes $\mathcal{X} \cap \overline{(\mathcal{Y} \cup \mathcal{Z})}$

This notation can cause major confusion when grading tests. Students often will have the overline going over parts of the expression, but in an ambiguous way. In other words, the overline doesn't start and stop in a reasonable position, so it is completely impossible to understand what the student has written. Consider

horrible notation \rightarrow $\mathcal{X} \overline{(\mathcal{Y} \cup \mathcal{Z})}$ or maybe $\mathcal{X} \cap (\mathcal{Y} \overline{\cup \mathcal{Z}})$ \leftarrow horrible notation

For this reason, I do not use the overline in class, or in my writings. **Students may not use the overline on tests or exams.**

Kenneth Rosen, who has written the famous encyclopedia *Discrete Mathematics and Its Applications* uses the overline, but many textbooks avoid it, as do my writings.

Minor Notation Notes

Unrelated to the above, Rosen uses the zero-with-a-slash-through-it to represent the empty set, instead of $\{\}$. As you noticed from Module 7.1: “Introduction to Set Theory,” I prefer $\{\}$. Again, most textbooks agree with me that $\{\}$ is a much clearer way of writing the empty set than a zero-with-a-slash-through-it.

Last but not least, I wrote the size of a set as $\#\mathcal{A}$. That's just the number of members of the set, which can be zero, any positive integer, or infinity. Some books write this as $|\mathcal{A}|$, using the same symbols as absolute value. This is not a mere question of writing. In some sense, the absolute value of a real number is the size of that real number. So it makes sense to use the absolute value symbol as the symbol for size. That's why the length of a line segment \overline{AB} in high school geometry is written $|\overline{AB}|$. Physicists use $|\vec{v}|$ to represent the magnitude (length) of the vector \vec{v} , which is another notion of size.

Questions:

Definition: At a few points in my textbook's Module 7.1: "Introduction to Set Theory," (namely 7-1-32 to 7-1-38), we made the point that a set with n members has 2^n subsets. (This count includes the improper subset and the trivial subset.) Related to this is the concept of taking *the power set* of a set. The power set of \mathcal{A} is the set of all subsets of \mathcal{A} . Sometimes we write it as $P(\mathcal{A})$, and sometimes we just say "the power set of \mathcal{A} ." For example, consider the set $\{1, 2, 3, 4\}$. Since we have 4 members, we expect $2^4 = 16$ subsets. The power set is

$$P(\{1, 2, 3, 4\}) = \{\{\}; \{1\}; \{2\}; \{3\}; \{4\}; \{1, 2\}; \{1, 3\}; \{1, 4\}; \{2, 3\}; \{2, 4\}; \{3, 4\}; \\ \{1, 2, 3\}; \{1, 2, 4\}; \{1, 3, 4\}; \{2, 3, 4\}; \{1, 2, 3, 4\}\}$$

Problem One: Tell me...

- What is $P(\{5, 6, 7\})$?
- What is $P(\{8, 9\})$?
- What is $P(\{0\})$?
- What is $P(\{\})$?

Investigate: If you look at the answers to the previous question, you'll see that indeed,

- The set $\{1, 2, 3, 4\}$ had $2^4 = 16$ subsets.
- The set $\{5, 6, 7\}$ had $2^3 = 8$ subsets.
- The set $\{8, 9\}$ had $2^2 = 4$ subsets.
- The set $\{0\}$ had $2^1 = 2$ subsets.
- The set $\{\}$ had $2^0 = 1$ subset, namely the empty set.
- Clearly, it really does seem to be the case that a set with n members has 2^n possible subsets, at least for $n < 5$.

Definition: For two sets \mathcal{A} and \mathcal{B} , the notation $\mathcal{A} - \mathcal{B}$ represents the set of things in \mathcal{A} that are not in \mathcal{B} . Consider

$$\mathcal{X} = \{1, 3, 5, 7\} \text{ and } \mathcal{Y} = \{1, 2, 3, 4\} \text{ as well as } \mathcal{Z} = \{4, 5, 6, 7\}$$

we would say that

$$\mathcal{X} - \mathcal{Y} = \{5, 7\}$$

Problem Two: With that definition in mind, tell me

- What is $\mathcal{X} - \mathcal{Z}$?
- What is $\mathcal{Y} - \mathcal{X}$?
- What is $\mathcal{Y} - \mathcal{Z}$?
- What is $\mathcal{Z} - \mathcal{X}$?
- What is $\mathcal{Z} - \mathcal{Y}$?
- What is $\mathcal{Z} - \mathcal{Z}$?

Note: By the way, it is worth noting that this operation is called *set subtraction*. Some older books will write $\mathcal{A} \setminus \mathcal{B}$ instead of $\mathcal{A} - \mathcal{B}$, but that notation has become rare.

Problem Three: I'd like to ask you a more general question about set subtraction. Is it true that

$$\mathcal{A} - \mathcal{B} = \mathcal{B} - \mathcal{A}$$

for all sets?

Hint: look at your answers for $\mathcal{X} - \mathcal{Z}$ and $\mathcal{Z} - \mathcal{X}$ from the previous question.

Problem Four: Think back to Module 7.3: “Intermediate Venn Diagram Problems,” where we shaded 2-circle Venn Diagrams with yellow and white, in order to show what various set-theoretic formulas meant. In that style, draw a Venn Diagram for $\mathcal{A} - \mathcal{B}$.

Definition: There is a minor but useful bit of notation called *the Cartesian Product*. When we write $\mathcal{A} \times \mathcal{B}$, we mean all possible ordered pairs, with the first member coming from \mathcal{A} and the second member coming from \mathcal{B} . For example

$$\{1, 2\} \times \{3, 4, 5\} = \{(1, 3); (1, 4); (1, 5); (2, 3); (2, 4); (2, 5)\}$$

and similarly

$$\{x, y, z\} \times \{a, b\} = \{(x, a); (x, b); (y, a); (y, b); (z, a); (z, b)\}$$

Note: Notice, we prohibit $(1, 2)$ in the first example, and (x, y) in the second example. That’s because $2 \notin \{3, 4, 5\}$ and $y \notin \{a, b\}$. Of course, $(3, 1)$ is completely out of the question in the first example, because $3 \notin \{1, 2\}$ and $1 \notin \{3, 4, 5\}$. This remains true even though $(1, 3)$ is a member of the first example.

Important: This reveals the real distinction between an ordered pair and a set. With ordered pairs

$$(1, 3) \neq (3, 1)$$

whereas with sets

$$\{1, 3\} = \{3, 1\}$$

because order does not matter inside a set.

If you keep this distinction in mind, then you’ll never forget when to use parentheses, e.g. $(1, 3)$, versus braces, e.g. $\{1, 3\}$. And while mathematical notation can be very strict at times, it is amazingly powerful. There’s a lot of depth involved, and the symbols carry a lot of hidden meaning. In this case, the choice of parentheses versus braces tells you whether order matters or order does not matter. Perhaps you now understand why professors enforce notation strictly.

About Semicolons: Whenever we have a set of sets, there is a convention. The small sets making up the set of sets will have commas separating the entries. The set of sets will have semicolons separating the small sets from each other. Though this might seem like a tedious distinction, it really helps. It is also good etiquette to put more of a space after each semicolon than after each comma. This also applies to sets of ordered pairs.

To illustrate the effectiveness of this convention, compare the following

$$\{x, y, z\} \times \{a, b\} = \{(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)\} \leftarrow \text{imperfect notation}$$

to what we had above, right after Problem Four, and think about which is easier to read.

Problem Five: Compute for me the following:

- (a) $\{0, 3\} \times \{1, 2, 4, 8\}$
- (b) $\{1, 2, 3, 4, 5\} \times \{0\}$
- (c) $\{a, b, c\} \times \{x, y, z\}$
- (d) $\{a, b\} \times \{b, c\}$

Investigate: You might or might not have noticed a pattern in the sizes of those Cartesian products.

- $\{x, y, z\} \times \{a, b\}$ had 6 members, $\#\{x, y, z\} = 3$, and $\#\{a, b\} = 2$.
- $\{1, 2, 3, 4, 5\} \times \{0\}$ had 5 members, $\#\{1, 2, 3, 4, 5\} = 5$, and $\#\{0\} = 1$.

- $\{0, 3\} \times \{1, 2, 4, 8\}$ had 8 members, $\#\{0, 3\} = 2$, and $\#\{1, 2, 4, 8\} = 4$.
- $\{a, b, c\} \times \{x, y, z\}$ had 9 members, $\#\{a, b, c\} = 3$, and $\#\{x, y, z\} = 3$.
- $\{a, b\} \times \{b, c\}$ had 4 members, $\#\{a, b\} = 2$, and $\#\{b, c\} = 2$.
- So we clearly know now that

$$\#(\mathcal{A} \times \mathcal{B}) = (\#\mathcal{A})(\#\mathcal{B})$$

which is just mathematical notation for “the size of the Cartesian product is the product of the sizes of the sets.”

- Therefore, it seems reasonable that the symbol chosen for the Cartesian product was \times .
- Yes, it is the same René Descartes (1596–1650) who invented the coordinate plane, bridging algebra and geometry, and who wrote the treatise about the evil genius (predicting virtual reality in the 1630s, roughly 3.5 centuries before it became available), as well as coming up with the famous quote “je pense, donc je suis” or “cogito ergo sum.” He was a busy guy!

Application: Surely the Cartesian product looks useless, or at best, unrelated to computer science. Guess again! It is part of a hugely important operation called “a cross join,” and it is used in relational databases. Suppose you have a database with a set of instructors, \mathcal{I} , and a set of classes \mathcal{C} . The course assignments for the coming term are just a subset of $\mathcal{I} \times \mathcal{C}$, in the sense that each course assignment is an ordered pair, connecting an instructor to a course. Likewise, **amazon.com** has lots of products, and most might be at one or two of many warehouses, while some might be at most warehouses. The database that tracks this will have a set for products, \mathcal{P} , and a set of warehouse \mathcal{W} . A subset of $\mathcal{P} \times \mathcal{W}$ will tell you which products are at which warehouses.

Only slightly related to this is the concept of an equivalence relation. It is one of our late topics in Math-270: *Discrete Mathematics*, and depends upon the idea of the Cartesian product.

Definition: When we say “the multiples of 3” we mean the set

$$\text{multiples}(3) = \{3, 6, 9, 12, 15, \dots\}$$

which, unless otherwise stated, is always in terms of the positive integers. As you can see, this is an infinite set. We cannot possibly list all the members of this set. Instead, it is customary to give five examples of the pattern of an infinite set, and then put some dots.

Problem Six: In similar notation to the previous paragraph, write out the following:

- multiples(5)
- multiples(2)

Definition: When we say “the common multiples of 4 and 6” we mean the set

$$\text{multiples}(4) \cap \text{multiples}(6) = \{4, 8, 12, 16, 20, \dots\} \cap \{6, 12, 18, 24, 30, \dots\} = \{12, 24, 36, 48, 60, \dots\}$$

Problem Seven: With the above definition in mind...

- Write out the common multiples of 2 and 5. (Hint, refer to the answers of the previous question.)
- Write out the common multiples of 6 and 8.
- Write out the common multiples of 3 and 12.

Investigate: Now that you’ve completed the previous question, you’ve probably noticed that every answer was the set of multiples of some number. Specifically,

- We noticed that $\text{multiples}(4) \cap \text{multiples}(6) = \text{multiples}(12)$.

- We noticed that $\text{multiples}(2) \cap \text{multiples}(5) = \text{multiples}(10)$.
- We noticed that $\text{multiples}(6) \cap \text{multiples}(8) = \text{multiples}(24)$.
- We noticed that $\text{multiples}(3) \cap \text{multiples}(12) = \text{multiples}(12)$.

With that in mind, we can save ourselves a lot of writing!

Definition: We should instead define the “least common multiple of x and y ,” abbreviated $\text{lcm}(x, y)$. Then we can write $\text{lcm}(4, 6) = 12$, $\text{lcm}(2, 5) = 10$, $\text{lcm}(6, 8) = 24$, and $\text{lcm}(3, 12) = 12$.

It’s worth mentioning that there are *much faster* ways of computing the lcm than writing out the sets of the multiples. We do not have time to discuss that here.

Problem Eight: Let’s say you’re helping a younger relative with adding fractions. The first step is to find the lowest common denominator.

- When adding $1/4 + 1/6$, what is the lowest common denominator?
- When adding $1/2 + 1/5$, what is the lowest common denominator?
- When subtracting $1/6 - 1/8$, what is the lowest common denominator?
- When adding $1/3 + 1/12$, what is the lowest common denominator?

Note: Unless you’re asleep, you’ve surely noticed that the “lowest common denominator” when adding (or subtracting) fractions is actually the same thing as the “least common multiple.” However, in university-level mathematics, we always say “least common multiple” and not “lowest common denominator.”

Application: Hopefully, at this point in your mathematical career, you’ve had the opportunity to solve a complicated math problem using a computer algebra package. While computer algebra packages tend to have almost all the same capabilities, a consequence of the fact that they are competing products, the “kingdom” of computer algebra packages is divided into two “phyla.” One phylum is the real-number based, floating-point style, as taught in MSCS-446: *Numerical Analysis* at UW Stout. Those algorithms are fast, but subject to rounding error. The other phylum is exact, integer-based, and slow, but totally immune to all rounding error. Those use exact rational arithmetic.

In other words, instead of storing $1/3$ as $0.3333333 \dots$, or more precisely $0.0101010101 \dots$ in binary, it is stored as 1 and 3 , or more precisely 00000001 and 00000011 in binary. Everything is represented as a fraction, if possible. As you do lots of additions and subtracts of fractions, you need to know a lot of lowest common denominators (or we should say, least common multiples).

Moreover, in order to work efficiently, you have to use certain tricks. Doing a long sequence of fraction additions and subtractions will often result in a huge numerator and denominator. Exact rational arithmetic was thought of soon after the invention of the computer in the 1940s. However, only in the last thirty years has it become feasible. That’s partly due to theoretical innovations, and partly because computers are a lot faster these days. I will say more after Problem Thirteen.

Problem Nine: Since the intersections of the sets of multiples had a meaning, you might be curious as to what happens when we take a union of sets of multiples. Referring to your work on the previous two questions, answer the following:

Note: Please write the first ten members of these sets, instead of the usual first five members.

- What is $\text{multiples}(4) \cup \text{multiples}(6)$?
- What is $\text{multiples}(6) \cup \text{multiples}(8)$?
- What is $\text{multiples}(2) \cup \text{multiples}(5)$?

Definition: For a particular number, it can be interesting to consider the set of divisors of that number. For example, suppose that I have ten eggs. I can divide them into 2 groups of 5 eggs, 5 groups of 2 eggs, 10 groups of 1 egg, and 1 group of 10 eggs. Therefore, we write $\text{divisors}(10) = \{1, 2, 5, 10\}$.

How is this computed? One way is to count upward, and simply rule in, or rule out, each integer. For example, if I ask for the divisors of 12, then you could perform the following mental process:

Does 1 divide 12? Yes. Does 2 divide 12? Yes. Does 3 divide 12? Yes.
 Does 4 divide 12? Yes. Does 5 divide 12? No. Does 6 divide 12? Yes.
 Does 7 divide 12? No. Does 8 divide 12? No. Does 9 divide 12? No.
 Does 10 divide 12? No. Does 11 divide 12? No. Does 12 divide 12? Yes.

Therefore, we write $\text{divisors}(12) = \{1, 2, 3, 4, 6, 12\}$.

Problem Ten: Compute for me now the divisors of 15.

Cool Trick: What if I ask for the divisors of 60? Do you really have to count up to 60? The answer is no—there is a faster way!

First, you count upward until you exceed the square root of the original number. Then you find the “mirror images” of the previous numbers. (The term “mirror image” is mathematical slang, and I’ll explain it in a moment. The technical term is co-factor.) For example, since $8 = \sqrt{64}$ then $8 > \sqrt{60} > 7$, I do not need to consider eight nor any higher number, though I should consider 7. Now I ask myself the following:

Does 1 divide 60? Yes. Does 2 divide 60? Yes. Does 3 divide 60? Yes.
 Does 4 divide 60? Yes. Does 5 divide 60? Yes. Does 6 divide 60? Yes.
 Does 7 divide 60? No.

At this point, I am halfway there. I have written down now

$$\text{divisors}(60) = \{1, 2, 3, 4, 5, 6,$$

I shall compute the “mirror images” as follows: $60 \div 6 = 10$, $60 \div 5 = 12$, $60 \div 4 = 15$, $60 \div 3 = 20$, $60 \div 2 = 30$, and $60 \div 1 = 60$. I can complete the divisor set with these new numbers, giving me the complete set of the divisors.

$$\text{divisors}(60) = \left\{ 1, 2, 3, 4, 5, 6, \underbrace{10, 12, 15, 20, 30, 60}_{\text{mirror images}} \right\}$$

As you can see, that’s much easier than doing 60 trial divisions.

Another Example: Just to make sure that the trick is clear, let me show it to you again, but for 42 this time. Since $6 = \sqrt{36}$ and $7 = \sqrt{49}$, then $7 > \sqrt{42} > 6$, so I will check 6 but not 7.

Does 1 divide 42? Yes. Does 2 divide 42? Yes. Does 3 divide 42? Yes.
 Does 4 divide 42? No. Does 5 divide 42? No. Does 6 divide 42? Yes.

At this point, I am halfway there. I have written down now

$$\text{divisors}(42) = \{1, 2, 3, 6,$$

I shall compute the “mirror images” as follows: $42 \div 6 = 7$, $42 \div 3 = 14$, $42 \div 2 = 21$, and $42 \div 1 = 42$. I can complete the divisors set with these new numbers, giving me the complete set of the divisors:

$$\text{divisors}(42) = \left\{ 1, 2, 3, 6, \underbrace{7, 14, 21, 42}_{\text{mirror images}} \right\}$$

Problem Eleven: Now try that cool trick yourself.

- (a) Compute the set of divisors of 36. Since $6 = \sqrt{36}$ thus $7 > \sqrt{36} \geq 6$, you will test 6 but not test 7.
- (b) Compute the set of divisors of 72. We know that $9 = \sqrt{81}$ and $8 = \sqrt{64}$, thus $9 > \sqrt{72} > 8$, therefore you will test 8 but not test 9.
- (c) Compute the set of divisors of 40.

Definition: At this point, you have the sets of divisors for 10, 12, 15, 40, 36, 60, and 72. We have another term to define. The “set of common divisors of 15 and 40” means

$$\text{divisors}(15) \cap \text{divisors}(40) = \{1, 3, 5, 15\} \cap \{1, 2, 4, 5, 8, 10, 20, 40\} = \{1, 5\}$$

Problem Twelve: Similarly, compute the following sets:

- (a) What is the set of common divisors of 36 and 40?
- (b) What is the set of common divisors of 36 and 60?
- (c) What is the set of common divisors of 72 and 60?
- (d) What is the set of common divisors of 40 and 60?
- (e) What is the set of common divisors of 36 and 42?
- (f) What is the set of common divisors of 40 and 42?
- (g) What is the set of common divisors of 42 and 72?

Problem Thirteen: Previously, we saw that we could write the set of common multiples of x and y as the set of multiples of some other number. You might be wondering if it is possible to rewrite the set of common divisors of x and y as the set of divisors of some other number. The answer is yes. Rewrite your answers to the previous question, each being the set of divisors of some number. I’ll do the first one for you.

Note: Don’t compute from scratch! You did 95% of the work for this problem during Problem Twelve.

- (a) What is the set of common divisors of 36 and 40? $\{1, 2, 4\} = \text{divisors}(4)$.
- (b) What is the set of common divisors of 36 and 60?
- (c) What is the set of common divisors of 72 and 60?
- (d) What is the set of common divisors of 40 and 60?
- (e) What is the set of common divisors of 36 and 42?
- (f) What is the set of common divisors of 40 and 42?
- (g) What is the set of common divisors of 42 and 72?

Definition: Just as we saw for the “least common multiple,” we can save ourselves a lot of writing here. (Efficiency in both mathematics and computer science is born of laziness, which is funny to me, because laziness is viewed as a *negative* attribute in high school, but becomes a *positive* attribute in later life.) If we want to represent the set of common divisors, we can simply report the greatest common divisor, which is abbreviated gcd. For example, we would say that the $\text{gcd}(36, 40) = 4$, while the $\text{gcd}(36, 42) = 6$. As with the lcm, there are much faster ways of computing the gcd than listing out the divisor set. One of them, the Extended Euclidean Algorithm, is very central in cryptography.

Application: You might be wondering what the gcd is used for. It looks kind of useless, perhaps? As it turns out, the opposite is true. First, the gcd is the “mother of all operations” in exact rational arithmetic. It is crucial, because that’s how you can algorithmically reduce fractions to lowest terms in one step.

For example, suppose that after some rational arithmetic, we get $72/120$. The bad way to approach this might look like the following:

$$\frac{72}{120} = \frac{36}{60} = \frac{18}{30} = \frac{9}{15} = \frac{3}{5}$$

because you just expended eight divisions.

Instead, we can compute $\gcd(72, 120) = 24$, as well as $72 \div 24 = 3$ and $120 \div 24 = 5$. That way, we know $72/120$ is really $3/5$. We only needed two divisions. When coding up exact rational arithmetic, you need to reduce to lowest terms really, really often. Sometimes it is done after each and every addition, subtraction, multiplication, or division. If you don’t do it so often, the the numerators and the denominators will quickly balloon into huge numbers.

Application: Diophantine equations are problems where we want to solve an equation, but we’re only interested in integer solutions. For degree one, this is a neat problem, not too hard to learn, popular in pure math competitions at both the high school and college level. It centers on Bezout’s theorem, which is itself centered on the gcd. We can even apply the gcd inside matrices, and find all integer solutions to multivariable linear systems of equations. That’s some lovely pure mathematics, but you might wonder if it has applications.

Application: There is a technique for balancing complicated chemical equations with matrices. It comes out of the pure math used to solve multivariate linear systems of Diophantine equations—though the chemists might be surprised to hear this. It uses the gcd, which is why I’m mentioning it. I wrote this up as a project for students, in Chapter 2.4 of my book *Sage for Undergraduates*, published by the American Mathematical Society in 2015. If you’ve had a course that deals with matrices, then you can read this, because that book is free in electronic form.

<http://www.gregorybard.com/books.html>

Application: Also, the gcd will be phenomenally important in number theory (a big topic in this course) and in particular, cryptography—the science of codes.

Problem Fourteen: Back in Problem Thirteen, you found seven gcds for me, before you knew what the acronym gcd meant. While the following looks pointless, it is really cool, trust me. Re-using those gcds, calculate the following for me.

- (a) What is $\frac{(36)(40)}{\gcd(36,40)}$?
- (b) What is $\frac{(36)(60)}{\gcd(36,60)}$?
- (c) What is $\frac{(72)(60)}{\gcd(72,60)}$?
- (d) What is $\frac{(40)(60)}{\gcd(40,60)}$?
- (e) What is $\frac{(36)(42)}{\gcd(36,42)}$?
- (f) What is $\frac{(40)(42)}{\gcd(40,42)}$?
- (g) What is $\frac{(42)(72)}{\gcd(42,72)}$?

Note: The following two are rather hard.

Problem Fifteen: Can you complete the following equation, in general?

$$\frac{(x)(y)}{\gcd(x, y)} = ??$$

Problem Sixteen: Looking back at the investigation right after Problem Five, can you come up with a similar formula for $\#(\mathcal{A} - \mathcal{B})$?

Answers:

1. Here are the power sets:

(a) $P(\{5, 6, 7\}) = \{\{\}; \{5\}; \{6\}; \{7\}; \{5, 6\}; \{5, 7\}; \{6, 7\}; \{5, 6, 7\}\}$

(b) $P(\{8, 9\}) = \{\{\}; \{8\}; \{9\}; \{8, 9\}\}$

(c) $P(\{0\}) = \{\{\}; \{0\}\}$

(d) $P(\{\}) = \{\{\}\}$

2. For the question that introduces set subtraction:

(a) What is $\mathcal{X} - \mathcal{Z}$? [Answer: $\{1, 3\}$.]

(b) What is $\mathcal{Y} - \mathcal{X}$? [Answer: $\{2, 4\}$.]

(c) What is $\mathcal{Y} - \mathcal{Z}$? [Answer: $\{1, 2, 3\}$.]

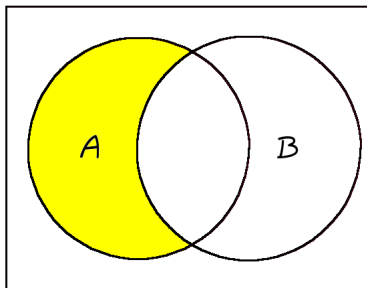
(d) What is $\mathcal{Z} - \mathcal{X}$? [Answer: $\{4, 6\}$.]

(e) What is $\mathcal{Z} - \mathcal{Y}$? [Answer: $\{5, 6, 7\}$.]

(f) What is $\mathcal{Z} - \mathcal{Z}$? [Answer: $\{\}$.]

3. Since in our example, we saw that $\mathcal{X} - \mathcal{Z} \neq \mathcal{Z} - \mathcal{X}$ then surely it is obvious that $\mathcal{A} - \mathcal{B} \neq \mathcal{B} - \mathcal{A}$ is not true in general, for all sets.

4. My diagram looks like this:



5. The solutions to the Cartesian Product question are

(a) $\{0, 3\} \times \{1, 2, 4, 8\} = \{(0, 1); (0, 2); (0, 4); (0, 8); (3, 1); (3, 2); (3, 4); (3, 8)\}$

(b) $\{1, 2, 3, 4, 5\} \times \{0\} = \{(1, 0); (2, 0); (3, 0); (4, 0); (5, 0)\}$

(c) $\{a, b, c\} \times \{x, y, z\} = \{(a, x); (a, y); (a, z); (b, x); (b, y); (b, z); (c, x); (c, y); (c, z)\}$

(d) $\{a, b\} \times \{b, c\} = \{(a, b); (a, c); (b, b); (b, c)\}$

6. The multiples are

(a) $\text{multiples}(5) = \{5, 10, 15, 20, 25, \dots\}$.

(b) $\text{multiples}(2) = \{2, 4, 6, 8, 10, \dots\}$.

7. The sets of common multiples are

(a) We noticed that $\text{multiples}(4) \cap \text{multiples}(6) = \{12, 24, 36, 48, 60, \dots\}$.

(b) We noticed that $\text{multiples}(2) \cap \text{multiples}(5) = \{10, 20, 30, 40, 50, \dots\}$.

(c) We noticed that $\text{multiples}(6) \cap \text{multiples}(8) = \{24, 48, 72, 96, 120, \dots\}$.

(d) We noticed that $\text{multiples}(3) \cap \text{multiples}(12) = \{12, 24, 36, 48, 60, \dots\}$.

8. The lowest common denominators are:

(a) When adding $1/4 + 1/6$, what is the lowest common denominator? [Answer: 12.]

(b) When adding $1/2 + 1/5$, what is the lowest common denominator? [Answer: 10.]

(c) When adding $1/6 + 1/8$, what is the lowest common denominator? [Answer: 24.]

(d) When adding $1/3 + 1/12$, what is the lowest common denominator? [Answer: 12.]

9. The unions of those sets of multiples are

(a) $\text{multiples}(4) \cup \text{multiples}(6) = \{4, 6, 8, 12, 16, 18, 20, 24, 28, 30, \dots\}$

(b) $\text{multiples}(6) \cup \text{multiples}(8) = \{6, 8, 12, 16, 18, 24, 30, 32, 36, 40, \dots\}$

(c) $\text{multiples}(2) \cup \text{multiples}(5) = \{2, 4, 5, 6, 8, 10, 12, 14, 15, 16, \dots\}$

Note: There is clearly some structure here, but the structure appears to be different in each case.

10. $\text{divisors}(15) = \{1, 3, 5, 15\}$.

11. Here are the sets of divisors:

(a) $\text{divisors}(36) = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$.

(b) $\text{divisors}(72) = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$.

(c) $\text{divisors}(40) = \{1, 2, 4, 5, 8, 10, 20, 40\}$.

12. Here are the sets of common divisors:

(a) What is the set of common divisors of 36 and 40?

$$\text{divisors}(36) \cap \text{divisors}(40) = \{1, 2, 3, 4, 6, 9, 12, 18, 36\} \cap \{1, 2, 4, 5, 8, 10, 20, 40\} = \{1, 2, 4\}$$

(b) What is the set of common divisors of 36 and 60?

$$\text{divisors}(36) \cap \text{divisors}(60) = \{1, 2, 3, 4, 6, 9, 12, 18, 36\} \cap \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\} = \{1, 2, 3, 4, 6, 12\}$$

(c) What is the set of common divisors of 72 and 60?

$$\text{divisors}(72) \cap \text{divisors}(60) = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\} \cap \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\} = \{1, 2, 3, 4, 6, 12\}$$

(d) What is the set of common divisors of 40 and 60?

$$\text{divisors}(40) \cap \text{divisors}(60) = \{1, 2, 4, 5, 8, 10, 20, 40\} \cap \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\} = \{1, 2, 4, 5, 10, 20\}$$

(e) What is the set of common divisors of 36 and 42?

$$\text{divisors}(36) \cap \text{divisors}(42) = \{1, 2, 3, 4, 6, 9, 12, 18, 36\} \cap \{1, 2, 3, 6, 7, 14, 21, 42\} = \{1, 2, 3, 6\}$$

(f) What is the set of common divisors of 40 and 42?

$$\text{divisors}(40) \cap \text{divisors}(42) = \{1, 2, 4, 5, 8, 10, 20, 40\} \cap \{1, 2, 3, 6, 7, 14, 21, 42\} = \{1, 2\}$$

(g) What is the set of common divisors of 42 and 72?

$$\text{divisors}(42) \cap \text{divisors}(72) = \{1, 2, 3, 6, 7, 14, 21, 42\} \cap \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\} = \{1, 2, 3, 6\}$$

13. Here are the rewritten sets of common divisors:

(a) What is the set of common divisors of 36 and 40? $\{1, 2, 4\} = \text{divisors}(4)$

(b) What is the set of common divisors of 36 and 60? $\{1, 2, 3, 4, 6, 12\} = \text{divisors}(12)$

(c) What is the set of common divisors of 72 and 60? $\{1, 2, 3, 4, 6, 12\} = \text{divisors}(12)$

(d) What is the set of common divisors of 40 and 60? $\{1, 2, 4, 5, 10, 20\} = \text{divisors}(20)$

(e) What is the set of common divisors of 36 and 42? $\{1, 2, 3, 6\} = \text{divisors}(6)$

(f) What is the set of common divisors of 40 and 42? $\{1, 2\} = \text{divisors}(2)$

(g) What is the set of common divisors of 42 and 72? $\{1, 2, 3, 6\} = \text{divisors}(6)$

14. We can easily compute:

(a) What is $\frac{(36)(40)}{\text{gcd}(36,40)} = \frac{(36)(40)}{4}$? 360.

(b) What is $\frac{(36)(60)}{\text{gcd}(36,60)} = \frac{(36)(60)}{12}$? 180.

(c) What is $\frac{(72)(60)}{\text{gcd}(72,60)} = \frac{(72)(60)}{12}$? 360.

(d) What is $\frac{(40)(60)}{\text{gcd}(40,60)} = \frac{(40)(60)}{20}$? 120.

(e) What is $\frac{(36)(42)}{\text{gcd}(36,42)} = \frac{(36)(42)}{6}$? 252.

(f) What is $\frac{(40)(42)}{\text{gcd}(40,42)} = \frac{(40)(42)}{2}$? 840.

(g) What is $\frac{(42)(72)}{\text{gcd}(42,72)} = \frac{(42)(72)}{6}$? 504.

15. In general,

$$\frac{(x)(y)}{\text{gcd}(x,y)} = \text{lcm}(x,y)$$

which is yet another reason why the gcd is “the mother of all operations” in exact rational arithmetic. If we can code a good, fast, and efficient gcd algorithm, then we get a good, fast, and efficient lcm algorithm by applying the above formula.

16. The formula clearly can't be

$$\#(\mathcal{A} - \mathcal{B}) = (\#\mathcal{A}) - (\#\mathcal{B})$$

because of simple examples like

$$\{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\} \quad \text{but} \quad 3 - 3 \neq 2$$

As it turns out, the answer is

$$\#(\mathcal{A} - \mathcal{B}) = \#\mathcal{A} - \#(\mathcal{A} \cap \mathcal{B})$$

This can be written as

$$\#(\mathcal{A} - \mathcal{B}) = \#(\mathcal{A} \cup \mathcal{B}) - \#\mathcal{B}$$