Module 8-7: Pascal's Triangle and the Binomial Theorem

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A Note about Notation

Just to recall, all of the following mean the same thing:

$$_7C_4 = \binom{7}{4} = C^7_4 = 7C4 = 35$$

and they are (all) pronounced "seven choose four."

Those should not be mistaken for

$$_7P_4 = \underbrace{(7)(6)(5)(4)}_{4\text{ items}} = 840$$

Practice Proofs

1. Give a proof (algebraic or combinatorial) of the fact that

$$\binom{n}{k} = \binom{n}{n-k}$$

2. Give a proof (algebraic or combinatorial) of the fact that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

which is called "Pascal's Identity."

3. Give a proof (algebraic or combinatorial) of the shortcut formula for computing

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

4. Give an algebraic proof of the shortcut formula for computing

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + \binom{n}{n-1} - \binom{n}{n}$$

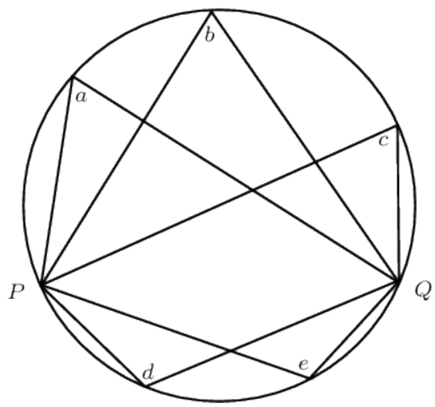
for odd n. I'll provide a combinatorial argument for you also.

Practice Problems

1. How many bit strings are there of length 20 with weight 5?
2. Compute the value of
$$\binom{16}{0}+\binom{16}{1}+\binom{16}{2}+\binom{16}{3}+\cdots+\binom{16}{15}+\binom{16}{16}$$
3. Compute the value of
$$\binom{16}{0}-\binom{16}{1}+\binom{16}{2}-\binom{16}{3}+\cdots-\binom{16}{15}+\binom{16}{16}$$
4. Compute the value of
$$1\binom{16}{0}+2\binom{16}{1}+4\binom{16}{2}+8\binom{16}{3}+16\binom{16}{4}+\cdots+2^{15}\binom{16}{15}+2^{16}\binom{16}{16}$$
5. In how many ways can a customer order a pizza, that has at least three toppings, if there are 12 possible toppings? Note: the order of the toppings doesn't matter. Toppings cannot be repeated—they are either present or absent. Further note: we do not rule out combinations that would be gross, such as pineapple with anchovies.
6. Repeat the previous problem, but permitting "at least two toppings" instead of "at least three."
7. What does Pascal's Triangle say that the expansion of $f(x) = (x+y)^6$ becomes, when written without parentheses/brackets?
8. What does Pascal's Triangle say that the expansion of $f(x) = (x+1)^5$ becomes, when written without parentheses/brackets?
9. What does Pascal's Triangle say that the expansion of $f(x) = (2x+3)^4$ becomes, when written without parentheses/brackets?
10. What is the coefficient of x^7y^{26} in $(x+y)^{33}$?
11. Compute some polynomials for me, that are shortcut formulas for...
    (a) $\ldots {}_xC_3$.
    (b) $\ldots {}_xC_2$.
    (c) $\ldots {}_xC_1$.
    (d) $\ldots {}_xC_0$.
12. Compute some polynomials for me, that are shortcut formulas for...
    (a) $\ldots {}_xC_x$.
    (b) $\ldots {}_xC_{x-1}$.
    (c) $\ldots {}_xC_{x-2}$.
    (d) $\ldots {}_xC_{x-3}$.

13. Suppose that I have chosen n distinct points scattered around the circumference of a circle. How many possible chords can I draw, connecting these? How many triangles can I draw, connecting these? Obviously, the answer depends on n. Please answer by giving me a polynomial in terms of n.

    In case you have forgotten what a chord is, here is a picture:

    ...but note that we haven't included all possible chords. We are missing *ab*, *ac*, *ae*, *ad*, *bc*, *be*, *bd*, *ce*, *cd*, QP, and *ed*. While there are 10 chords drawn in the image, these 11 missing chords would bring us to a total of 21 chords, for the case of 7 points.

14. Give me a catalog of how many subsets there are in a set of size 6. In other words, how many subsets are there of size 0? of size 1? of size 2? of size 3? of size 4? of size 5? of size 6? How many subsets in total?

15. If you happen to know that ${}_xC_y = a$, ${}_xC_{y-1} = b$, and ${}_xC_{y+1} = c$, then...

    - What is ${}_xC_{x-y}$?
    - What is ${}_xC_{x-y+1}$?
    - What is ${}_{x+1}C_{x-y+1}$?

My Proofs

1. This is an unusual problem, in that the algebraic proof is easier than the combinatorial. It is almost always the other way around.

   Algebraic Proof: Observe,

   $$\binom{n}{n-a} = \frac{n!}{(n-a)!(n-(n-a))!} = \frac{n!}{(n-a)!(n-n+a)!} = \frac{n!}{(n-a)!a!} = \binom{n}{a} \qquad \blacksquare$$

   Combinatorial Argument: Strictly speaking, this is a casual argument and not a formal proof.

   Suppose I have n objects, and I have to choose a subset of size x. I could either choose x objects to include in the subset, or I could choose $n-x$ objects to leave out of the subset. There are ${}_nC_x$ ways to do the former, and ${}_nC_{n-x}$ ways to do the latter. Yet the number of ways to do the former must equal the number of possible subsets, as must the number of ways to do the latter.

   Therefore,

   $${}_nC_x =_n C_{n-x}$$

   Even more casually, the following three numbers are equal:

- The number of ways to choose x things to include in the subset.
- The number of possible subsets.
- The number of ways to choose $n-x$ things to exclude from the subset.

2. Here we can see how an algebraic proof can be very ugly, but a combinatorial proof can be beautiful.

Algebraic Proof: Observe,

$$\begin{aligned}
\binom{n-1}{x}+\binom{n-1}{x-1} &= \frac{(n-1)!}{x!(n-1-x)!}+\frac{(n-1)!}{(x-1)!(n-1-(x-1))!} \\
&= \frac{(n-1)!}{x!(n-1-x)!}+\frac{(n-1)!}{(x-1)!(n-1-x+1)!} \\
&= \frac{(n-1)!}{x!(n-x-1)!}+\frac{(n-1)!}{(x-1)!(n-x)!} \\
&= \frac{(n-1)!(n-x)}{x!(n-x-1)!(n-x)}+\frac{x(n-1)!}{x(x-1)!(n-x)!} \\
&= \frac{(n-1)!(n-x)}{x!(n-x)!}+\frac{x(n-1)!}{x!(n-x)!} \\
&= \frac{(n-1)!}{x!(n-x)!}[(n-x)+x] \\
&= \frac{(n-1)!}{x!(n-x)!}(n) \\
&= \frac{n(n-1)!}{x!(n-x)!} \\
&= \frac{n!}{x!(n-x)!} \\
&= \binom{n}{x}
\end{aligned}$$

Therefore,

$$\binom{n-1}{x}+\binom{n-1}{x-1}=\binom{n}{x} \qquad \blacksquare$$

Combinatorial Proof: Suppose I have a set of n objects, and I want to choose a subset of size x. By definition, there are ${}_nC_x$ ways to do that. Designate one of the n members as "privileged," and the other $n-1$ as "unprivileged." Either the subset I create contains the privileged member, or it does not contain the privileged member.

If the subset that I create does not contain the privileged member, then I am choosing x objects from the $n-1$ unprivileged members. There are ${}_{n-1}C_x$ ways to do that.

If the subset that I create does contain the privileged member, then I am choosing $x-1$ objects from the $n-1$ unprivileged members. There are ${}_{n-1}C_{x-1}$ ways to do that.

Therefore, we obtain

$$\binom{n}{x}=\binom{n-1}{x}+\binom{n-1}{x-1} \qquad \blacksquare$$

3. Our goal is to prove that the sum is 2^n. More precisely,

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n-1}+\binom{n}{n}=2^n$$

Algebraic Proof: Observe, using Pascal's Triangle on $(1+1)^n$, we obtain

$$(-1+1)^n =_n C_0(1)^0(1)^n +_n C_1(1)^1(1)^{n-1} +_n C_2(1)^2(1)^{n-2} +_n C_3(1)^3(1)^{n-3} + \cdots +_n C_{n-1}(1)^{n-1}(1)^1 +_n C_n(1)^n(1)^0$$

which simplifies to

$$(1+1)^n =_n C_0(1)(1) +_n C_1(1)(1) +_n C_2(1)(1) +_n C_3(1)(1) + \cdots +_n C_{n-1}(1)(1) +_n C_n(1)(1)$$

and then

$$(1+1)^n =_n C_0 +_n C_1 +_n C_2 +_n C_3 + \cdots +_n C_{n-1} +_n C_n$$

Clearly, the right-hand side is the desired sum. Even more clearly, the left-hand side is

$$(1+1)^n = 2^n$$

and therefore,

$$2^n =_n C_0 +_n C_1 +_n C_2 +_n C_3 + \cdots +_n C_{n-1} +_n C_n \qquad \blacksquare$$

Combinatorial Argument: (This is casual, and so should be considered "an argument" and not a "proof.") Consider all n-bit binary strings. Surely order matters, because 1101101 is a different string than 1111100. Moreover, repeats are allowed, because I just repeated several 0s and 1s. Since order matters and repeats are allowed, we are using the exponent principle. The size of the alphabet is 2, and the string is length n, giving us 2^n possible binary strings.

Yet, we could also classify those strings by their weight. The weight of a binary string is the number of entries that happen to be ones. A binary string of length n can have weight 0, weight 1, weight 2, weight 3, ..., weight $n-1$, or weight n.

For any integer weight $0 \leq w \leq n$, there are ${}_nC_w$ ways to to construct a binary string of length n and weight w. If we add all of those up, we get the total number of n-bit binary strings, which we know to be 2^n. Therefore,

$${}_nC_0 +_n C_1 +_n C_2 +_n C_3 + \cdots +_n C_{n-1} +_n C_n = 2^n$$

4. Now we're told to construct an algebraic proof only. Yet, I've provided a combinatorial argument, too.

Algebraic Proof: Observe, using Pascal's Triangle on $(-1+1)^n$, we obtain

$$(-1+1)^n =_n C_0(-1)^0(1)^n +_n C_1(-1)^1(1)^{n-1} +_n C_2(-1)^2(1)^{n-2} +_n C_3(-1)^3(1)^{n-3} + \cdots +_n C_{n-1}(-1)^{n-1}(1)^1 +_n C_n(-1)^n(1)^0$$

which simplifies to

$$(-1+1)^n =_n C_0(1)(1) +_n C_1(-1)(1) +_n C_2(1)(1) +_n C_3(-1)(1) + \cdots +_n C_{n-1}(1)(1) +_n C_n(-1)(1)$$

and then

$$(-1+1)^n =_n C_0 -_n C_1 +_n C_2 -_n C_3 + \cdots +_n C_{n-1} -_n C_n$$

Clearly, the right-hand side is the desired alternating sum. Even more clearly, the left-hand side is

$$(-1+1)^n = 0^n = 0$$

and therefore,

$$0 =_n C_0 -_n C_1 +_n C_2 -_n C_3 + \cdots +_n C_{n-1} -_n C_n \qquad \blacksquare$$

Combinatorial Proof: Consider all n-bit binary strings. The weight of a binary string is the number of entries that happen to be ones. How many have even weight? That would be the number of binary strings with 0 ones, 2 ones, 4 ones, 6 ones, ..., $n-1$ ones, all added together.

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots + \binom{n}{n-1}$$

How many have odd weight? That would be the number of binary strings with 1 one, 3 ones, 5 ones, 7 ones, ..., n ones, all added together.

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \binom{n}{7} + \cdots + \binom{n}{n}$$

These totals should be the same, because I can always take a string, and flip all the 1s to 0s and 0s to 1s. For all n, weight w becomes weight $n-w$. For odd n, the even-weight strings become odd-weight, and the odd-weight strings become even-weight. Therefore

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots + \binom{n}{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \binom{n}{7} + \cdots + \binom{n}{n}$$

Observe that subtracting the right hand side, we get

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \binom{n}{5} + \cdots + \binom{n}{n-1} - \binom{n}{n} = 0 \quad \blacksquare$$

Solutions to Practice Problems

1. There are ${}_20C_5$ bit strings of weight 5 and length 20. That's because we have 20 slots for 0s and 1s, and we have to choose 5 slots from among those 20, where the 1s will go. (Of course, you can also choose 15 slots from among those 20 where the 0s will go.) Either way, you get

$$\binom{20}{5} = \binom{20}{15} = 15,504$$

2. This is the sum of a row of Pascal's triangle. It is Row 16, because the top entry is 16 in each case. Therefore, the answer is 2^{16}. Another way to look at this is that you are finding *all possible* subsets of a set of size 16. That's called "the power set."

3. This is clearly an alternating sum of a row of Pascal's triangle. Therefore, the answer is zero.

4. This is what we called "the weighted sum." The answer is 3^{16}.

5. With x being the answer that we want, we can think of this question as:

$$\binom{12}{3} + \binom{12}{4} + \binom{12}{5} + \binom{12}{6} + \cdots + \binom{12}{11} + \binom{12}{12} = x$$

but then we can add the three "missing terms" to each side. We obtain

$$\binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \binom{12}{3} + \cdots + \binom{12}{11} + \binom{12}{12} = \binom{12}{0} + \binom{12}{1} + \binom{12}{2} + x$$

The left hand-side is clearly the sum of the 12th row of Pascal's Triangle, so that is $2^{12} = 4096$. Then we can plug in the others, because ${}_{12}C_0 = 1$, ${}_{12}C_1 = 12$, and ${}_{12}C_2 = 66$. We get

$$4096 = 1 + 12 + 66 + x$$

and finally

$$x = 4096 - 1 - 12 - 66 = 4017$$

6. This is the same thing as the previous problem, but without the ${}_{12}C_2 = 66$ term. We get 4083.

7. We see that Row # 6 is "1, 6, 15, 20, 15, 6, 1" and therefore we write

$$(x+y)^6 = 1x^6y^0 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1x^0y^6$$

but of course, we can remove the 0th powers, and the coefficients of one. Now we get

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

8. We see that Row # 5 is "1, 5, 10, 10, 5, 1" and therefore we write

$$(x+1)^5 = 1x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1x^0y^5$$

and again, we can remove the 0th powers, all powers of 1, and the coefficients of one. Cleaning it we get

$$(x+1)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

9. We see that Row # 4 is "1, 4, 6, 4, 1" and therefore we write

$$(2x+3)^4 = 1(2x)^4 3^0 + 4(2x)^3 3^1 + 6(2x)^2 3^2 + 4(2x)^1 3^3 + 1(2x)^0 3^4$$

and then do some arithmetic, obtaining

$$(2x+3)^4 = 1(16x^4)1 + 4(8x^3)3 + 6(4x^2)9 + 4(2x)27 + 1(1)81$$

which becomes

$$(2x+3)^4 = 16x^4 + 96x^3 + 216x^2 + 216x + 81$$

10. What is the coefficient of x^7y^{26} in $(x+y)^{33}$? Of course, it is just ${}_{33}C_7 = 4,272,048$.

If you like, you can verify this in Sage:

```
var("x y")

f(x,y) = (x+y)^33

f.expand()
```

You have to scroll around until you find the x^7y^{26} term. Indeed, it has coefficient 4,272,048.

11. These are computed as follows:

(a) We have

$$\binom{x}{3} = \frac{x!}{(x-3)!3!} = \frac{x!}{(x-3)!} \cdot \frac{1}{3!} = {}_xP_3 \cdot \frac{1}{6} = \frac{(x)(x-1)(x-2)}{6}$$

(b) We have

$$\binom{x}{2} = \frac{x!}{(x-2)!2!} = \frac{x!}{(x-2)!} \cdot \frac{1}{2!} = {}_xP_2 \cdot \frac{1}{2} = \frac{(x)(x-1)}{2}$$

(c) Remember, anything "choose 1" is just itself, so the answer is x. That's how we named the rows of Pascal's Triangle.

(d) Remember, anything "choose 0" is just one. That's because the empty set is unique. Another way to look at this is to see that the left-edge of Pascal's Triangle is all ones.

12. Because of symmetry, this is actually the previous problem in disguise.

(a) ${}_xC_x = {}_xC_0 = 1$

(b) ${}_xC_{x-1} = {}_xC_1 = x$

(c) ${}_xC_{x-2} = {}_xC_2 = \frac{(x)(x-1)}{2}$ as computed earlier.

(d) ${}_xC_{x-3} = {}_xC_3 = \frac{(x)(x-1)(x-2)}{6}$ as computed earlier.

13. If you pick a chord, you are picking two endpoints. You cannot connect point c to point c, so repeats are not allowed. Connecting a to b and connecting b to a gives the same chord, therefore, order does not matter. Since repeats are not allowed and order does not matter, we are using the combinations principle. The answer is ${}_nC_2$ but we were asked for a polynomial. Looking two problems ago, we see that

$$\binom{x}{2} = \frac{(x)(x-1)}{2}$$

Similarly, if you pick a triangle, you are picking three vertices. You cannot connect point c, point c, and point c, nor can you connect point a to point b to point a. Therefore, repeats are not allowed. Connecting any of the following

$$\{abc, acb, bac, bca, cab, cba\}$$

will create the same triangle, so order does not matter. Since repeats are not allowed and order does not matter, we are using the combinations principle, as before. The answer is ${}_nC_3$ but we were asked for a polynomial. Again, looking two problems ago, we see that

$$\binom{x}{3} = \frac{(x)(x-1)(x-2)}{6}$$

14. This is just the 6th row of Pascal's Triangle: "1, 6, 15, 20, 15, 6, 1."

- There is 1 subset of size six. It is sometimes called the "improper" subset.
- There are 6 subsets of size five.
- There are 15 subsets of size four.
- There are 20 subsets of size three.
- There are 15 subsets of size two.
- There are 6 subsets of size one.
- There is 1 subset of size zero. That is the empty set, sometimes called the "trivial" subset.
- If we add these up, we get the total number of subsets of a set of size 6. That's totaling the 6th row of Pascal's Triangle. Therefore, we know the total will be $2^6 = 64$ without adding it up. Naturally, if you do add them up, you get the right answer anyway:

$$1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$$

15. If you happen to know that ${}_xC_y = a$, ${}_xC_{y-1} = b$, and ${}_xC_{y+1} = c$, then...

- What is ${}_xC_{x-y}$? a
- What is ${}_xC_{x-y+1}$? b
- What is ${}_{x+1}C_{x-y+1}$? $a + b$

Note: If the previous problem is mysterious, let's check with $x = 17$ and $y = 5$, two arbitrarily chosen specific values. (Of course, this is not a proof.)

- ${}_xC_y = {}_{17}C_5 = 6188 = a$

- ${}_xC_{y-1} ={}_{17} C_4 = 2380 = b$
- ${}_xC_{y+1} ={}_{17} C_6 = 12,376 = c$
- ${}_xC_{x-y} ={}_{17} C_{12} = 6188 = a$
- ${}_xC_{x-y+1} ={}_{17} C_{13} = 2380 = b$
- ${}_{x+1}C_{x-y+1} ={}_{18} C_{13} = 8568 = 6188 + 2380 = a + b ={}_{17} C_{12} +{}_{17} C_{13}$