Pascal's Triangle Reference Sheet

Math-270: Discrete Mathematics

November 16, 2015

The Triangle Itself:

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

What Makes this Work?

$$\text{Pascal's Identity: } \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

The Four Cool Properties:

$$\text{The Symmetry Property: } \binom{n}{k} = \binom{n}{n-k}$$

$$\text{The Power-Set Property: } \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n-1} + \binom{n}{n} = \sum_{j=0}^{j=n} \binom{n}{j} = 2^n$$

$$\text{The Alternating-Sum Property: } \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \mp \binom{n}{n-1} \pm \binom{n}{n} = \sum_{j=0}^{j=n} (-1)^j \binom{n}{j} = 0$$

$$\text{The Weighted-Sum Property: } 1\binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + 8\binom{n}{3} + 16\binom{n}{4} + 32\binom{n}{5} + \cdots + 2^n\binom{n}{n} = \sum_{j=0}^{j=n} 2^j \binom{n}{j} = 3^j$$

$$
\begin{aligned}
(1)({}_0C_0) &= \\
(1)(1) &= \\
1 &= 3^0
\end{aligned}
$$

$$
\begin{aligned}
(1)({}_1C_0) + (2)({}_1C_1) &= \\
(1)(1) + (2)(1) &= \\
1 + 2 &= 3 = 3^1
\end{aligned}
$$

$$
\begin{aligned}
(1)({}_2C_0) + (2)({}_2C_1) + (4)({}_2C_2) &= \\
(1)(1) + (2)(2) + (4)(1) &= \\
1 + 4 + 4 &= 9 = 3^2
\end{aligned}
$$

$$
\begin{aligned}
(1)({}_3C_0) + (2)({}_3C_1) + (4)({}_3C_2) + (8)({}_3C_3) &= \\
(1)(1) + (2)(3) + (4)(3) + (8)(1) &= \\
1 + 6 + 12 + 8 &= 27 = 3^3
\end{aligned}
$$

$$
\begin{aligned}
(1)({}_4C_0) + (2)({}_4C_1) + (4)({}_4C_2) + (8)({}_4C_3) + (16)({}_4C_4) &= \\
(1)(1) + (2)(4) + (4)(6) + (8)(4) + (16)(1) &= \\
1 + 8 + 24 + 32 + 16 &= 81 = 3^4
\end{aligned}
$$

$$
\begin{aligned}
(1)({}_5C_0) + (2)({}_5C_1) + (4)({}_5C_2) + (8)({}_5C_3) + (16)({}_5C_4) + (32)({}_5C_5) &= \\
(1)(1) + (2)(5) + (4)(10) + (8)(10) + (16)(5) + (32)(1) &= \\
1 + 10 + 40 + 80 + 80 + 32 &= 243 = 3^5
\end{aligned}
$$

$$
\begin{aligned}
(1)({}_6C_0) + (2)({}_6C_1) + (4)({}_6C_2) + (8)({}_6C_3) + (16)({}_6C_4) + (32)({}_6C_5) + (64)({}_6C_6) &= \\
(1)(1) + (2)(6) + (4)(15) + (8)(20) + (16)(15) + (32)(6) + (64)(1) &= \\
1 + 12 + 60 + 160 + 240 + 192 + 64 &= 729 = 3^6
\end{aligned}
$$

$$
\begin{aligned}
(1)({}_7C_0) + (2)({}_7C_1) + (4)({}_7C_2) + (8)({}_7C_3) + (16)({}_7C_4) + (32)({}_7C_5) + (64)({}_7C_6) + (128)({}_7C_7) &= \\
(1)(1) + (2)(7) + (4)(21) + (8)(35) + (16)(35) + (32)(21) + (64)(7) + (128)(1) &= \\
1 + 14 + 84 + 280 + 560 + 672 + 448 + 128 &= 2187 = 3^7
\end{aligned}
$$