Mini-Worksheet: Conglomerates, Holding Companies, and Matrix Algebra

Gregory V. Bard

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Overview

Here we're going to analyze a conglomerate that manufactures various products but which has fairly questionable practices. The conglomerate has 10 manufacturing plants, scattered throughout The Rust Belt, and each is worth a different amount. However, in order to hide the total worth of the conglomerate and in order to make themselves harder to sue in a lawsuit, the ownership is masked through four different holding companies. This might be important if the conglomerate is breaking environmental regulations. Such deception also would make them very hard to tax, but luckily the IRS has people on staff who know mathematics—including matrix algebra.

Now we're going to see how a typical knot of ownership would be constructed, and how matrix algebra can untangle it.

Assets

1. Gary, Indiana — 4.6 million.
2. Flint, Michigan — 2.9 million.
3. Detroit, Michigan — 5.1 million.
4. Cleveland, Ohio — 3.9 million.
5. Buffalo, New York — 1.7 million.
6. Scranton, Pennsylvania — 4.2 million.
7. Pittsburg, Pennsylvania — 9.6 million.
8. Cincinnati, Ohio — 5.4 million.
9. Camden, New Jersey — 2.7 million.
10. Baltimore, Maryland — 6.3 million.

Note: Total Tangible Assets — 46.4 million.

Ownership Model

- Company Alpha owns the Gary, Cleveland, and Scranton plants, plus 20% of Company Beta, 30% of Company Gamma and 10% of Company Delta.
- Company Beta owns 25% of Company Alpha, 15% of Company Gamma, and 30% of Company Delta, as well as the Flint, Camden and Baltimore plants.
- Company Gamma owns 40% of Company Alpha, 45% of Company Beta, but none of Company Delta, yet it owns the Detroit and Cincinnati plants.
- Company Delta owns 15% of Company Alpha, 30% of Company Beta, and 25% of Company Gamma, and the plants in Buffalo as well as Pittsburg.

We must now compute how much each company is worth.

The Equations

Let a be the value of Company Alpha, b be the value of Company Beta, g be the value of Company Gamma, and d be the value of Company Delta.

$$\begin{aligned} a &= 4.6 + 3.9 + 4.2 + 0.2b + 0.3g + 0.1d \\ b &= 0.25a + 0.15g + 0.3d + 2.9 + 2.7 + 6.3 \\ c &= 0.4a + 0.45b + 0d + 5.1 + 5.4 \\ d &= 0.15a + 0.3b + 0.25g + 1.7 + 9.6 \end{aligned}$$

Sadly, these are not in Standard Form. In standard form, they become

$$\begin{aligned} a - 0.2b - 0.3g - 0.1d &= 12.7 \\ -0.25a + b - 0.15g - 0.3d &= 11.9 \\ -0.4a - 0.45b + g - 0d &= 10.5 \\ -0.15a - 0.3b - 0.25g + d &= 11.3 \end{aligned}$$

Looks like a Problem that a Matrix Could Solve

Then we can construct a matrix:

$$\left[\begin{array}{cccc|c} 1 & -0.2 & -0.3 & -0.1 & 12.7 \\ -0.25 & 1 & -0.15 & -0.3 & 11.9 \\ -0.4 & -0.45 & 1 & 0 & 10.5 \\ -0.15 & -0.3 & -0.25 & 1 & 11.3 \end{array}\right]$$

Time to Wake up Sage

With that in mind, I type the following into Sage, at the usual URL (`https://sagecell.sagemath.org/`).

```
A = matrix( 4, 5, [1, -0.2, -0.3, -0.1, 12.7, -0.25, 1, -0.15, -0.3, 11.9, -0.4, -0.45, 1, 0,
10.5, -0.15, -0.3, -0.25, 1, 11.3 ] )

print "Question:"
print A
print "Answer:"
print A.rref()
```

Here is the response from Sage:

```
Question:
[  1.00000000000000 -0.200000000000000 -0.300000000000000 -0.100000000000000   12.7000000000000]
[-0.250000000000000   1.00000000000000 -0.150000000000000 -0.300000000000000   11.9000000000000]
[-0.400000000000000 -0.450000000000000   1.00000000000000  0.000000000000000   10.5000000000000]
[-0.150000000000000 -0.300000000000000 -0.250000000000000   1.00000000000000   11.3000000000000]
Answer:
[     1.00000000000000 -2.77555756156289e-17     0.000000000000000     0.000000000000000      37.6613308941230]
[    0.000000000000000      1.00000000000000     0.000000000000000     0.000000000000000      39.7553167162131]
[    0.000000000000000 -1.11022302462516e-16      1.00000000000000     0.000000000000000      43.4544248799451]
[    0.000000000000000 -5.55111512312578e-17     0.000000000000000      1.00000000000000      39.7394008689687]
Powered by Sage
```

Checking the Work

But we aren't done yet—we have to check our work!

$$\begin{aligned}
a &= 4.6 + 3.9 + 4.2 + 0.2(39.7553) + 0.3(43.4544) + 0.1(39.7394) = 37.66132 \quad \leftarrow \texttt{:)} \\
b &= 0.25(37.6613) + 0.15(43.4544) + 0.3(39.7394) + 2.9 + 2.7 + 6.3 = 39.755305 \quad \leftarrow \texttt{:)} \\
c &= 0.4(37.6613) + 0.45(39.7553) + 0(39.7394) + 5.1 + 5.4 = 43.454405 \quad \leftarrow \texttt{:)} \\
d &= 0.15(37.6613) + 0.3(39.7553) + 0.25(43.4544) + 1.7 + 9.6 = 39.739385 \quad \leftarrow \texttt{:)}
\end{aligned}$$

Since all equations are satisfied to six or more significant figures, then we know we've done our work correctly. (Actually, even five digits would have been enough.) However, if we had used nine digits in our intermediate calculations, then we'd match even better.

One Last Thought

Now here's a mystery... the total assets have a value of 46.4 million dollars. Yet

$$37.6613 + 39.7553 + 43.4544 + 39.7394 = 160.6104$$

and so the sum of the values of the holding companies comes to 160.6 million dollars!

This is how companies can appear to be more valuable on paper than they actually are.

Try it Yourself

Now consider a conglomerate with 3 holding companies. Company Kappa owns 25% of Company Lambda, a factory worth 13 million, and 20% of Company Omicron, and a foundry worth 10 million. Company Lambda owns a factory worth 21 million, 30% of Company Kappa, and 50% of Company Omicron. Company Omicron owns 35% of Company Kappa, 15% of Company Lambda, and an assembly plant worth 25 million. How much is each holding company worth? What are the total tangible assets? How much are the three companies worth collectively?

(On the last page of this mini-worksheet, the raw system of equations is provided, not in standard form, along with the final answer.)

The Answers to the Try-It-Yourself

Let x be the value of Company Kappa, let y be the value of Company Lambda, and let z be the value of Company Omicron.

$$\begin{aligned} x &= 0.25y + 13 + 0.20z + 10 \\ y &= 21 + 0.3x + 0.5z \\ z &= 0.35x + 0.15y + 25 \end{aligned}$$

Finally, we will eventually conclude that

- Company Kappa is worth $ 48,511,515.98.
- Company Lambda is worth $ 61,127,535.23.
- Company Omicron is worth $ 51,148,160.88.
- The total tangible assets come to $ 69,000,000.
- The holding companies together are worth $ 160,787,212.09 on paper, however.

Now we can check our work with...

$$\begin{aligned} (0.25)(61,127,535.23) + 13,000,000 + (0.20)(51,148,160.88) + 10,000,000 &= \\ 15,281,883.80 + 13,000,000 + 10,229,632.17 + 10,000,000 &= 48,511,515.97 \end{aligned}$$

$$\begin{aligned} 21,000,000 + (0.30)(48,511,515.98) + (0.50)(51,148,160.88) &= \\ 21,000,000 + 14,553,454.79 + 25,574,080.44 &= 61,127,535.23 \end{aligned}$$

$$\begin{aligned} (0.35)(48,511,515.98) + (0.15)(61,127,535.23) + 25,000,000 &= \\ 16,979,030.59 + 9,169,130.28 + 25,000,000 &= 51,148,160.87 \end{aligned}$$

As you can see, the first and last are off by one penny, and the middle by a dollar, so we can be confident in the accuracy of our work.