An Algebra-Based Approach to Leontief Input-Output Analysis

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1 Leontief Input-Output Analysis

In this brief document we'll explore a bit about an important macroeconomic model. The purpose of the model is to see how different sectors of the economy affect each other, and how the consumption of one commodity to make another commodity drives demand. While still in active use, the model is from the 1928 doctoral dissertation of Wassily Wassilyovich Leontief (1906–1999). It is notable to point out that the first non-military use of computers was the solution of a problem of this type: dividing the US economy into over 500 sectors—easily one of the largest single computations ever performed by anyone until that time. The results were rather accurate predictions. The model is not only popular in capitalist societies, but was also heavily used in the former Soviet Union and in the People's Republic of China. Today it is used primarily by state and provincial governments, but also in foreign aid projects.

Of course, we will use technology to solve our problems, even though they are much smaller than those of Leontief's model of the US economy. I think at the end of this exploration you'll have a large appreciation for why this topic was basically untouched after its discovery in 1928 until the invention of the computer, yet you'll also understand why it has been an extremely popular topic from the invention of the computer to the present era.

2 The Opening Example

In this problem we'll model the economy of the coal industry of a highly industrialized region, such as parts of China or West Virginia. Coal is produced from mines, but railways are required to transport the ore to the refinery, which runs on electricity. Power plants burn the coal to make electricity, but they need railways to bring them the coal. The railroads run partly on electricity and partly on coal to deliver these resources. While there is no consumer demand nor external demand for the rail network connecting the coal mines, refineries, and power plants, there certainly is lots of demand for electric power among the commercial, residential and other industrial sectors. There is also some external industrial demand for coal—for example, in making steel out of iron.

Suppose that producing a dollar's worth of coal requires 0.15 dollars of electricity, 0.30 dollars of railways, but no coal. Also suppose that making a dollar of electricity requires 0.25 dollars of coal, 0.05 dollars of electricity (to run the plant), and 0.20 dollars of railways. Furthermore suppose that a dollar of rail transport requires 0.25 dollars of coal, 0.10 dollars of electricity, and 0.05 dollars of rail transport (e.g. repositioning, repair, and depot maintenance). Finally suppose that the external demand for coal is worth 10 million dollars, the external demand for electricity is worth 100 million dollars, but that there is no external demand for these particular rail lines.

How much (in dollars) should be produced of coal, electricity, and railroads?

3 Is the Solution Obvious?

Before we proceed, we should note that the answer isn't obvious. As noted, the demand is for 10 million dollars of coal and 100 million dollars of electricity. Just for the electricity demand, we would consume 25 million dollars of coal, 5 million dollars of electricity, and 20 million dollars of rails. This consumption during production is called *internal consumption*, to distinguish it from *external demand*.

On our second attempt, we could guess 35 million dollars of coal, 105 million dollars of electricity, and 20 million dollars of rail transport. The second guess is constructed by adding the internal consumption of each resource to its external demand. However, careful analysis of the situation indicates that we have failed again:

- Producing 35 million dollars of coal requires $(35 \times 0.15) = 5.25$ million dollars of electricity, and $(35 \times 0.30) = 10.5$ million dollars of rail transport.
- The production of 20 million dollars of rail transport requires $(20 \times 0.25) = 5$ million dollars of coal, $(20 \times 0.10) = 2$ million in electricity, and $(20 \times 0.05) = 1$ million dollars of rails.
- The 105 million dollars of electricity required $(105 \times 0.25) = 26.25$ million in coal, $(105 \times 0.05) = 5.25$ million in electricity, and $(105 \times 0.20) = 21$ million in rail transport.

How close did our second attempt get?

- Out of 105 million dollars of electricity produced, 5.25 million was consumed by the coal industry, 2 million by the rail industry and 5.25 internally consumed in the production of electricity. This leaves $105 - 5.25 - 2 - 5.25 = 92.5$ million for external demand, a shortfall of 7.5 million dollars.
- Of the 35 million dollars of coal produced, 5 million went to the rail industry, and 26.25 million went to the electricity industry. This leaves $35 - 5 - 26.25 = 3.75$ million worth of coal supplied for external demand, a terrible shortfall of 6.25 million dollars from 10 million dollars of actual demand.
- Then the rail industry has a demand of 10.5 million from the coal industry, a demand of 1 million from itself (repositioning, maintenance, and repair), and 21 million from the electricity industry (bringing coal to the power plants for burning to produce electricity). The demand is 32.5 million but we only produced 20 million, which is a disastrous shortfall.

In summary, what we have here is a tangled knot. Changing the production quantity of one item will change the internal consumption of nearly every other industry. In other words, changing anything results in changing everything. It is amazing that this kind of problem can be solved at all. With guess-and-check, a patient person can actually solve this specific three-industry problem with a great deal of time invested. Another interesting problem that we'll see later has six industries instead of three, and that probably cannot be solved by hand—at least not without an extraordinary amount of effort. Leontief's famous model of the US economy in 1946 had over 500 industries. However, with matrix algebra we can make quick work of any problem of this type, so long as we are willing to input the required matrix.

4 Experimentation

At this stage, you can experiment with an interactive webpage ("an App") that will allow you to move sliders to produce various amounts of coal, electricity, and rails. You can try to meet both internal and external demand.

1. Go to `http://www.gregorybard.com/`
2. Click on "Interactive Webpages for Teaching Math."
3. Click on "Leontief Input-Output Analysis."
4. Follow the on screen directions.

5 Algebraic Solution to Our Opening Problem

Let x signify the amount (in dollars) of coal produced; likewise y for electricity, and z for railways.

We have to produce 10 million dollars of coal to satisfy external demand, plus the extra that is consumed by the coal, electricity, and rail industries. Therefore we write

$$x = \underbrace{10,000,000}_{\text{ex. demand}} + \underbrace{0x}_{\text{coal}} + \underbrace{0.25y}_{\text{elec.}} + \underbrace{0.25z}_{\text{rails}}$$

Likewise, we have to produce 100 million dollars of electricity to satisfy external demand, and similarly the extra that is consumed by the coal, electricity, and rail industries. Accordingly we write

$$y = \underbrace{100,000,000}_{\text{ex. demand}} + \underbrace{0.15x}_{\text{coal}} + \underbrace{0.05y}_{\text{elec.}} + \underbrace{0.10z}_{\text{rails}}$$

Last but not least, we need not produce any dollars of railway costs, except that which must be consumed by the coal, electricity, and rail industries. For that we write

$$z = \underbrace{0}_{\text{ex. demand}} + \underbrace{0.30x}_{\text{coal}} + \underbrace{0.20y}_{\text{elec.}} + \underbrace{0.05z}_{\text{rails}}$$

We now have the following system of equations:

$$\begin{aligned} x &= 10,000,000 + 0x + 0.25y + 0.25z \\ y &= 100,000,000 + 0.15x + 0.05y + 0.10z \\ z &= 0 + 0.30x + 0.20y + 0.05z \end{aligned}$$

This system of equations is not in standard form. To put it in standard form, we need all the variables on the left. Moving all the variables to the left and combining like terms results in

$$\begin{aligned} x - 0.25y - 0.25z &= 10,000,000 \\ -0.15x + 0.95y - 0.10z &= 100,000,000 \\ -0.30x - 0.20y + 0.95z &= 0 \end{aligned}$$

As a matrix, this would be written as

$$\left[\begin{array}{ccc|r} 1 & -0.25 & -0.25 & 10,000,000 \\ -0.15 & 0.95 & -0.10 & 100,000,000 \\ -0.30 & -0.20 & 0.95 & 0 \end{array}\right]$$

According to Sage (the commands will be given below), the RREF (Reduced Row Echelon Form) of this matrix turns out to be

$$\left[\begin{array}{ccc|r} 1 & 0 & 0 & 49,400,164.\cdots \\ 0 & 1 & 0 & 117,304,847.\cdots \\ 0 & 0 & 1 & 40,295,809.\cdots \end{array}\right]$$

Thus the solution to the linear system of equations is $x = 49,400,164$; $y = 117,304,847$; and $z = 40,295,809$. That tells us the industry must produce \$ 49,400,164 of coal, \$ 117,304,847 worth of electricty, and \$ 40,295,809 in railway transport.

6 The Sage Commands Used

It is probably worth noting the Sage commands that I used. They are

```
A = matrix( 3, 4, [1, -0.25, -0.25, 10, -0.15, 0.95, -0.1, 100, -0.3, -0.2, 0.95, 0] )

print "Question:"
print A

print "Answer:"
print A.rref()
```

The first thing to point out is that when defining the matrix `A`, I must tell Sage to use three rows and four columns. Then I enter the four entries of the first row of the matrix, followed by the four entries of the second row of the matrix, and finally the four entries of the third row of the matrix. Be absolutely certain to place the commas, parentheses and brackets precisely where I have placed them. As you can see, I represented $ 10,000,000 and $ 100,000,000 as 10 and 100, respectively. This means that the unit of money which I was using is "millions of dollars."

Meanwhile, the `print "Question:"` and `print "Answer:"` commands are purely decorative, serving only to make the output more human-readable. The `print A` command is there to help me verify that I entered the matrix into the computer correctly. (Actually, as it turns out, I did not enter it correctly the first time. Instead of positive `0.95` in the third row, I had typed `-0.95`. Therefore, you can see that it is very important to check whether you typed your matrix into the computer correctly, by seeing if the question matrix matches the matrix you had desired to type in.)

Here is the output which I have obtained:

```
Question:
[  1.00000000000000 -0.250000000000000 -0.250000000000000   10.0000000000000]
[-0.150000000000000  0.950000000000000 -0.100000000000000   100.000000000000]
[-0.300000000000000 -0.200000000000000  0.950000000000000  0.000000000000000]
Answer:
[     1.00000000000000 -2.77555756156289e-17 -5.55111512312578e-17      49.4001643385374]
[    0.000000000000000      1.00000000000000 -2.77555756156289e-17      117.304847986853]
[    0.000000000000000 -5.55111512312578e-17      1.00000000000000      40.2958093672966]
```

At first we might panic and think that the matrix comprising the left 3 columns of the output is not the identity matrix; of course, we seek the identity matrix in those 9 entries. Actually, what we must realize is that the `e-17` means 10^{-17}, which we can all agree is really close to zero.

A quintillionth is actually 10^{-18}, so 10^{-17} is ten quintillionths (or 10 millionths of a trillionth). This is one percent of one quadrillionth. Since our unit is millions of dollars, 10^{-6} is one dollar, 10^{-9} is a tenth of a cent, 10^{-12} is one 10,000th of a cent, 10^{-15} is one ten millionth of a cent, and 10^{-17} is one billionth of a penny—definitely negligible.

7 Checking Our (and Sage's) Work

We'll check our work industry-by-industry. First we'll check the coal industry:

- The coal industry requires $\$ 49,400,164 \times 0 = \$ 0$ dollars of coal.
- The electricity industry requires $\$ 117,304,847 \times 0.25 = \$ 29,326,211$ dollars of coal.
- The rails industry requires $\$ 40,295,809 \times 0.25 = \$ 10,073,952$ dollars of coal.
- The consumers (and other industries not part of the model) require $ 10,000,000 of coal.

- The total demand for coal is $ \$ 0 + \$ 29,326,211 + \$ 10,073,952 + \$ 10,000,000 = \$ 49,400,163 $, only off by a dollar!

Next we'll check the electricity industry:

- The coal industry requires $ \$ 49,400,164 \times 0.15 = \$ 7,410,024 $ dollars of electricity.
- The electricity industry requires $ \$ 117,304,847 \times 0.05 = \$ 5,865,242 $ dollars of electricity.
- The rails industry requires $ \$ 40,295,809 \times 0.10 = \$ 4,029,580 $ dollars of electricity.
- The consumers (and other industries not part of the model) require $ 100,000,000 of electricity.
- The total demand for electricity is $ \$ 7,410,024 + \$ 5,865,242 + \$ 4,029,580 + \$ 100,000,000 = \$ 117,304,846 $, again—only off by a dollar!

Finally we'll check the railway industry:

- The coal industry requires $ \$ 49,400,164 \times 0.30 = \$ 14,820,049 $ dollars of rail transport.
- The electricity industry requires $ \$ 117,304,847 \times 0.20 = \$ 23,460,969 $ dollars of rail transport.
- The rails industry requires $ \$ 40,295,809 \times 0.05 = \$ 2,014,790 $ dollars of rail transport.
- The consumers (and other industries not part of the model) require $ 0 of these rail lines. After all, why would a consumer want to take a trip on a freight line connecting a coal mine with a refinery, or the lines connecting a refinery with a power plant?
- The total demand for rail transport is $ \$ 14,820,049 + \$ 23,460,969 + \$ 2,014,790 + \$ 0 = \$ 40,295,808 $, within a single dollar!

As you can see, everything has worked out splendidly. An expert would say that "we have checked our results by computing the internal consumptions and seeing that they match the computer-predicted production values and user-provided demand values." That means that the internal consumption plus the external demand equals the production.

8 A Note about Excessive Precision

Of course, had I chosen to use more precision in copying the numbers from Sage into this document, then I could get answers that match to the penny. However, knowing the demand is 100,000,000 dollars really means that the demand is between 99 and 101 million dollars. Most econometric statistics, especially at the macroeconomic level, can only be known to $\pm 1\%$ or $\pm 0.1\%$ in most cases—with the exception being analyses performed by a government, who have access to corporate tax returns and other sources of data which are unavailable to anyone else. Even then, the accuracy would be plus or minus one basis point. Therefore it is silly to go even to the thousands of dollars or hundreds of dollars, let alone to the penny.

Nonetheless, I perform my calculations to the dollar to better enable checking my work and/or grading.

9 Procedural Summary

In general, Input-Output Models can be built according to the following nine steps:

1. Slowly read the problem several times.
2. With the greatest of care, construct the system of equations.
3. Manipulate the equations into standard form. (Meaning, move all the variables to one side.)

4. Convert this system of equations into a matrix.
5. Enter the matrix into a computer.
6. Take a moment to check whether you've entered that matrix correctly and accurately.
7. Ask the computer to compute the RREF of that matrix.
8. Write down the computer-provided solution.
9. Check your answers by computing all the internal consumptions, and see if they sum to (including the external demand) the computer-provided production values.

$$\text{Remember:} \qquad \text{production} = \text{external demand} + \text{internal consumption}.$$

10 More Exercises

In the problems that follow I've provided you with the equations (before they are converted into standard form), the original matrix, and the final answer. These represent steps 2, 4, and 8 from the above list of steps.

Exercise One: A Modification to the Original Problem

Let us now suppose that a newly improved generator becomes available, and one can produce a dollar of electricity using only 20 cents of coal, instead of 25 cents of coal. Everything else remains the same. How much now (of coal, electricity, and rail transport) must be produced?

By the way, this raises an important point: while this method is great for short-term analysis, the coefficients, like 0.25, will eventually change over time as costs fluctuate or as technology becomes more efficient. Therefore, this method of Input-Output Analysis is not useful for long-term predictions.

Exercise Two: A Small Emirate

Here is a realistic model of a small semi-industrialized nation that mostly produces crude oil and gasoline, but also rice for local consumption. Naturally, all those industries are very dependent upon trucking. Few citizens have cars and thus rely on buses for transportation—which we fold into trucking to simplify the model. The citizens probably have a diverse diet but we model it here as consisting only of rice, which would be the staple ingredient in any case.

- Two cents of oil is needed to produce a dollar of oil (for lubricating the equipment); fifty cents of oil is needed to produce a dollar of gasoline; three cents of oil is needed for each dollar of truck transport (again, for lubrication); no oil is used in producing either rice or labor.
- Ten cents of gasoline is needed to produce a dollar of oil (for powering the on-site generators); no gasoline is used in the production of gasoline; forty cents of gasoline is used to produce a dollar of truck transport; fifteen cents of gasoline is used in producing rice (to power the tractors); no gasoline is used in producing labor.
- Ten cents of trucking is used to produce each dollar of oil; twenty cents of trucking is used to produce each dollar of gasoline; one cent of trucking is used to produce each dollar of trucking (representing the delivery of spare parts); thirty cents of trucking is used to produce a dollar of rice; twenty-five cents of trucking is used to produce a dollar of labor.
- No rice is used in making oil, nor gasoline, nor trucking. About five cents of rice is used as seed for next year's rice; finally, forty-five cents of rice go into each dollar of labor.

- Twenty cents of labor goes into each dollar of oil, and fifteen cents into each dollar of gasoline; each dollar of trucking requires twenty-five cents of labor; a dollar of rice requires twenty cents of labor; also each dollar of labor requires 25 cents of labor, in the form of cafeteria workers, janitors, guards, supervisors, and clerks.
- For simplicity, please use v for oil, w for gasoline; x for trucking; y for rice; and z for labor.
- The external demand is 5 billion dollars for oil, 250 million dollars of gasoline; 500 million dollars of trucking; 750 million dollars of rice. There is no external demand for labor.

You can see that we've got a fairly realistic model here, but we've totally neglected electricity and other utilities, as well as housing. That's why most models need to have seven or more variables to get a decent picture of an economy.

Exercise Three: an Alaskan Fishery

Here we will model the economy of a small fishery located in Alaska.

- Each dollar of raw tuna that is fished requires 19 cents of labor and 17 cents of shipping operations.
- Each dollar of canned tuna requires 40 cents of raw tuna, 8 cents of fuel, 15 cents of labor, and 10 cents of facilities services (e.g. buildings, cleaning, etc. . .).
- Each dollar of labor requires 25 cents of facilities services (mostly housing but also recreation spaces), 20 cents of food (1/4th of which is raw tuna, cooked fresh in the cafeteria, and 3/4th of which is brought in by shipping operations), and 9 cents of labor (supervisors, payroll clerks, security, etc. . .).
- Each dollar of facilities services requires 20 cents of labor, 15 cents of fuel (to heat the buildings), 5 cents of facilities services (sheds to store equipment and supplies) and 5 cents of shipping operations (to bring in cleaning supplies and other sundry items).
- Each dollar of fuel requires 40 cents of shipping operations (since it must be shipped in), 10 cents of facilities services (fuel tanks to store it in, which need to be maintained and heated) and 7 cents of labor (for loading, unloading, and pumping).
- Each dollar of shipping operations requires 20 cents of fuel, 15 cents of labor (the ship's crew), 10 cents of canned tuna (the crew's food), and 5 cents of facilities services (for dry dock and maintenance).
- There is an external demand of 50 million dollars of canned tuna, but nothing else.
- Let raw tuna be x_1, canned tuna be x_2, labor be x_3, facilities services be x_4, fuel be x_5, and shipping operations be x_6.

Exercise Four: Tourism Comes to Alaska

At this point, make sure you have completed Exercise Three, and compare your work with the answers in this worksheet. Now suppose that Alaska becomes a popular tourism destination. Represent the increase in tourism as an external demand of \$ 250,000 on shipping operations. It is interesting to see what impact this will have on the five other sectors of the fishery's economy. Redo the problem with this change, and note not only the final production values of each sector, but also how they changed (by how much they increased) from Exercise Three to Exercise Four.

Exercise Five: California before the Internet Boom

In this problem, and the next, we're going to model the economy of a small county in Southern California before and after the internet boom. The computer industry was already important in the 1980s, but wine and agriculture were the main activities outside of Hollywood. We can imagine dividing the economy into six sectors: labor, agriculture, public services, transportation, utilities & energy, computers & e-commerce. These can be x_1, x_2, ..., x_6.

- Each dollar of labor requires 30 cents of public services, 11 cents of agriculture (i.e. food), 5 cents of utilities & energy, 10 cents of transportation, 2 cents of computers & e-commerce, and 8 cents of labor (e.g. supervisors, secretaries, and so forth).
- Each dollar of agriculture requires 55 cents of labor, 20 cents of transportation, 10 cents of agriculture (e.g. seed, manure, animal feed), 3 cents of utilities & energy, and 2 cents of public services (e.g. USDA, meat inspectors, FDA, and so forth).
- Each dollar of public services requires 40 cents of labor, 5 cents of utilities & energy, 4 cents of computers & e-commerce, 1 cent of transportation and 9 cents of public services (e.g. internal affairs, budgetary operations, building operations).
- Each dollar of transportation requires 25 cents of labor, 1 cent of computers and e-commerce, 15 cents in utilities & energy, 1 cent of public services (e.g. USDOT, the ICC, toll collectors, traffic cops), 5 cents of transportation (i.e. repositioning and repairs).
- Each dollar of utilities & energy requires 35 cents of labor, 15 cents of transportation, 10 cents of public services, 2 cents of computers & e-commerce, 12 cents of utilities & energy (e.g. water to power the steam in an electricity plant, electricity in a water treatment plant, *et cetera*).
- Each dollar of computers & e-commerce requires 1 cent of transportation, 10 cents of utilities & energy (mostly electricity), 15 cents of computers and e-commerce (i.e. the backend: servers, routers, and IT infrastructure), and 65 cents of labor—after all, programmers are very expensive.
- There is an external demand of 100 million dollars in agriculture, but only 10 million dollars in computers & e-commerce.

Exercise Six: California after the Internet Boom

Now repeat the previous problem, but with the computer industry being 1 billion dollars.

11 Solutions to Exercises

Solutions to Exercise One:

The system of equations would become

$$\begin{aligned} x &= 10,000,000 + 0x + 0.20y + 0.25z \\ y &= 100,000,000 + 0.15x + 0.05y + 0.10z \\ z &= 0 + 0.30x + 0.20y + 0.05z \end{aligned}$$

The matrix would become

$$\left[\begin{array}{ccc|r} 1 & -0.20 & -0.25 & 10,000,000 \\ -0.15 & 0.95 & -0.10 & 100,000,000 \\ -0.30 & -0.20 & 0.95 & 0 \end{array}\right]$$

The final answer would become \$ 42,671,433 worth of coal; 115,989,600 worth of electricity; \$ 37,894,052 worth of rail transport.

Solutions to Exercise Two:

$$\begin{aligned} v &= 5000 + 0.02v + 0.50w + 0.03x + 0y + 0z \\ w &= 250 + 0.1v + 0w + 0.40x + 0.15y + 0z \\ x &= 500 + 0.1v + 0.20w + 0.01x + 0.30y + 0.25z \\ y &= 750 + 0v + 0w + 0x + 0.05y + 0.45z \\ z &= 0 + 0.2v + 0.15w + 0.25x + 0.20y + 0.25z \end{aligned}$$

$$\left[\begin{array}{rrrrr|r} 0.98 & -0.50 & -0.03 & 0 & 0 & 5000 \\ -0.1 & 1 & -0.4 & -0.15 & 0 & 250 \\ -0.1 & -0.2 & 0.99 & -0.3 & -0.25 & 500 \\ 0 & 0 & 0 & 0.95 & -0.45 & 750 \\ -0.2 & -0.15 & -0.25 & -0.2 & 0.75 & 0 \end{array}\right]$$

```
A = matrix( 5, 6, [0.98, -0.5, -0.03, 0, 0, 5000, -0.1, 1, -0.4, -0.15, 0, 250, -0.1, -0.2,
0.99, -0.3, -0.25, 500, 0, 0, 0, 0.95, -0.45, 750, -0.2, -0.15, -0.25, -0.2, 0.75, 0] )

print "Question:"
print A
print "Answer:"
print A.rref()
```

The final answer, after inputting the above Sage commands, is

- $ 6,655,428,913 dollars of oil,
- $ 2,822,516,089 dollars of gasoline,
- $ 3,702,076,352 dollars of trucking,
- $ 2,840,951,048 dollars of rice,
- $ 4,330,896,658 dollars of labor.
- Realistically, these numbers should be rounded off to the nearest million, but I'm displaying more digits of precision to help you check your work. When using mathematics in the workplace, it is important not to use excess precision; if someone does, then that someone is claiming a degree of precision that does not actually exist.

The total economy of the emirate is worth $ 20,351,869,030 or slightly above 20 billion dollars. Their economy is supported by those exports which only total $ 6.5 billion. As you can see, the internal consumption is an important part of the economy.

Solutions to Exercise Three:

$$\begin{aligned} x_1 &= 0 + 0.4x_2 + 0.05x_3 \\ x_2 &= 50 + 0.1x_6 \\ x_3 &= 0 + 0.19x_1 + 0.15x_2 + 0.09x_3 + 0.2x_4 + 0.07x_5 + 0.15x_6 \\ x_4 &= 0 + 0.1x_2 + 0.25x_3 + 0.05x_4 + 0.1x_5 + 0.05x_6 \\ x_5 &= 0 + 0.08x_2 + 0.15x_4 + 0.2x_6 \\ x_6 &= 0 + 0.17x_1 + 0.15x_3 + 0.05x_4 + 0.4x_5 \end{aligned}$$

$$\left[\begin{array}{rrrrrr|r} 1 & -0.4 & -0.05 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -0.1 & 50 \\ -0.19 & -0.15 & 0.91 & -0.2 & -0.07 & -0.15 & 0 \\ 0 & -0.1 & -0.25 & 0.95 & -0.1 & -0.05 & 0 \\ 0 & -0.08 & 0 & -0.15 & 1 & -0.2 & 0 \\ -0.17 & 0 & -0.15 & -0.05 & -0.4 & 1 & 0 \end{array}\right]$$

The final production totals of the fishery are as follows:

- \$ 21,275,538 of raw tuna.
- \$ 50,992,466 of canned tuna.
- \$ 17,571,034 of labor.
- \$ 11,331,199 of fuel.
- \$ 7,764,009 of facilities services.
- \$ 9,924,660 of shipping operations.

Solutions to Exercise Four:

The system of equations would be the same, except for the shipping operations equation (the last equation). The new equation for shipping operations would be

$$x_6 = 0.25 + 0.17x_1 + 0.15x_3 + 0.05x_4 + 0.4x_5$$

where the first term in the sum is now 0.25; that term used to be a zero.

The matrix would be the same, except that we would have 0.25 instead of 0 in the lower right-hand corner position. The new production totals of the fishery are as follows:

- \$ 21,290,794 of raw tuna (an increase of \$ 15,256).
- \$ 51,021,713 of canned tuna (an increase of \$ 29,247).
- \$ 17,642,176 of labor (an increase of \$ 71,142).
- \$ 11,375,495 of fuel (an increase of \$ 44,296).
- \$ 7,831,487 of facilities services (an increase of \$ 67,478).
- \$ 10,217,131 of shipping operations (an increase of \$ 292,471).

Solutions to Exercise Five:

- Let x_1 signify labor.
- Let x_2 signify agriculture (including the wine industry).
- Let x_3 signify public services.
- Let x_4 signify transportation.
- Let x_5 signify utilities and energy.
- Let x_6 signify computers and e-commerce.

$$
\begin{aligned}
x_1 &= 0.08x_1 + 0.55x_2 + 0.4x_3 + 0.25x_4 + 0.35x_5 + 0.65x_6 \\
x_2 &= 100,000,000 + 0.11x_1 + 0.1x_2 \\
x_3 &= 0.3x_1 + 0.02x_2 + 0.09x_3 + 0.01x_4 + 0.1x_5 \\
x_4 &= 0.1x_1 + 0.2x_2 + 0.01x_3 + 0.05x_4 + 0.15x_5 + 0.01x_6 \\
x_5 &= 0.05x_1 + 0.03x_2 + 0.05x_3 + 0.15x_4 + 0.12x_5 + 0.1x_6 \\
x_6 &= 10,000,000 + 0.02x_1 + 0.02x_2 + 0.04x_3 + 0.01x_4 + 0.02x_5 + 0.15x_6
\end{aligned}
$$

$$
\left[\begin{array}{rrrrrr|r}
0.92 & -0.55 & -0.4 & -0.25 & -0.35 & -0.65 & 0 \\
-0.11 & 0.90 & 0 & 0 & 0 & 0 & 100,000,000 \\
-0.3 & -0.02 & 0.91 & -0.01 & -0.1 & 0 & 0 \\
-0.1 & -0.2 & -0.01 & 0.95 & -0.15 & -0.01 & 0 \\
-0.05 & -0.03 & -0.05 & -0.15 & 0.88 & -0.1 & 0 \\
-0.02 & -0.02 & -0.04 & -0.01 & -0.02 & 0.85 & 10,000,000
\end{array}\right]
$$

- There is $ 135,679,181.38 of labor.
- There is $ 127,694,122.16 of agriculture and wine.
- There is $ 50,811,334.07 of public services.
- There is $ 45,907,496.53 of transportation.
- There is $ 25,216,021.81 of utilities and energy.
- there is $ 21,486,252.74 of computers and e-commerce.
- The total economy is $ 406,794,408.69.

Solutions to Exercise Six:

The variables are the same as in the previous exercise.

$$
\begin{aligned}
x_1 &= 0.08x_1 + 0.55x_2 + 0.4x_3 + 0.25x_4 + 0.35x_5 + 0.65x_6 \\
x_2 &= 100,000,000 + 0.11x_1 + 0.1x_2 \\
x_3 &= 0.3x_1 + 0.02x_2 + 0.09x_3 + 0.01x_4 + 0.1x_5 \\
x_4 &= 0.1x_1 + 0.2x_2 + 0.01x_3 + 0.05x_4 + 0.15x_5 + 0.01x_6 \\
x_5 &= 0.05x_1 + 0.03x_2 + 0.05x_3 + 0.15x_4 + 0.12x_5 + 0.1x_6 \\
x_6 &= 1,000,000,000 + 0.02x_1 + 0.02x_2 + 0.04x_3 + 0.01x_4 + 0.02x_5 + 0.15x_6
\end{aligned}
$$

$$
\left[\begin{array}{rrrrrr|r}
0.92 & -0.55 & -0.4 & -0.25 & -0.35 & -0.65 & 0 \\
-0.11 & 0.90 & 0 & 0 & 0 & 0 & 100,000,000 \\
-0.3 & -0.02 & 0.91 & -0.01 & -0.1 & 0 & 0 \\
-0.1 & -0.2 & -0.01 & 0.95 & -0.15 & -0.01 & 0 \\
-0.05 & -0.03 & -0.05 & -0.15 & 0.88 & -0.1 & 0 \\
-0.02 & -0.02 & -0.04 & -0.01 & -0.02 & 0.85 & 1,000,000,000
\end{array}\right]
$$

- $ 1,494,603,354.57
- $ 293,784,854.44

- $ 537,200,554.30
- $ 288,113,818.83
- $ 317,146,627.71
- $ 1,254,682,184.81
- The total economy is $ 4,185,531,394.66.
- That's a growth factor of 10.2890×.